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STANCU MODIFIED OPERATORS REVISITED

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Abstract. In this paper we construct a general positive approximation process representing an integral form in Kantorovich sense of the Stancu operators. By using K-functionals and some moduli of smoothness we give direct theorems for pointwise approximation. Also, by using the contraction principle we reobtain the convergence of the iterates of Stancu polynomials.

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1. INTRODUCTION

It is well known that the Stancu operators [10] are defined by

$$(1)\qquad (S_n^{(\alpha)}f)(x) := \sum_{k=0}^{n} w_{n,k}^{(\alpha)}(x) f\left(\tfrac{k}{n}\right), \quad f \in C[0,1],\ x\in[0,1],$$

where $w_{n,k}^{(\alpha)}(x) := \binom{n}{k} x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}/1^{[n,-\alpha]}$, $k=\overline{0,n}$, represent *the fundamental polynomials of Stancu* of n degree. Here $y^{[m,-\alpha]}$ stands for the generalized factorial power with the step $-\alpha$, $y^{[0,-\alpha]} := 1$ and $y^{[m,-\alpha]} := y(y+\alpha)\ldots(y+(m-1)\alpha)$, $m\in\mathbb{N}$.

Under the hypotheses that α is a non-negative real parameter depending on the natural number n and $\alpha=\alpha_n\to 0$ as $n\to\infty$, D.D. Stancu proved that the sequence $(S_n^{(\alpha)})_{n\ge 1}$ converges to the identity operator on the space $C[0,1]$. We keep this assumption throughout the paper.

In 1989 Quasim Razi [8] modified the operator $S_n^{(\alpha)}$ into integral form as follows

$$(2)\qquad (K_n^{(\alpha)}f)(x) := (n+1)\sum_{k=0}^{n} w_{n,k}^{(\alpha)}(x)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\mathrm{d}t, \quad x\in[0,1],$$

and f belongs to the space of real-valued integrable functions $L_1[0,1]$.

Further approximation properties were examined in [3] and [1].

The present paper focuses on two approaches. Firstly we generalize the operators defined by (2) and we study their degree of approximation in the terms both of the weighted Totik-Ditzian modulus of smoothness and the

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integral moduli of high order. Secondly, coming back to the operators $S_n^{(\alpha)}$ we reobtain the convergence of the iterates by using a new proof based on the contraction principle. This way it results that Stancu operators are weakly Picard operators.

2. THE OPERATORS $\mathcal{K}_n^{(\alpha)}$, $n \in \mathbb{N}$

We consider two real sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ verifying the following conditions

$$b_n \geq n+1, \quad a_n \leq 1, \quad n \in \mathbb{N}, \quad \text{and} \quad \inf_{n\in\mathbb{N}} a_n > 0. \tag{3}$$

For every f belonging to $L_1[0,1]$ we define the operators

$$(\mathcal{K}_n^{(\alpha)} f)(x) \equiv (\mathcal{K}_n^{(\alpha,a_n,b_n)} f)(x) := b_n \sum_{k=0}^{n} w_{n,k}^{(\alpha)}(a_n x) \int_{k/b_n}^{(k+1)/b_n} f(t)\mathrm{d}t, \tag{4}$$

where $x \in [0,1]$ and $n \in \mathbb{N}$.

Remarks. (i) The operators $\mathcal{K}_n^{(\alpha,a_n,b_n)}$, $n \in \mathbb{N}$, are linear. Since the sequences $(\alpha_n), (a_n), (b_n)$ are positive, the operators are positive too and consequently they become monotone.

(ii) In the particular case $a_n = 1$ and $b_n = n+1$ we reobtain the operator $K_n^{(\alpha)}$ defined by (2) and consequently $\mathcal{K}_n^{(0,1,n+1)}$ is the n^{th} classical Kantorovich operator. □

In what follows we denote by e_j the Korovkin test functions, $e_j(x) = x^j$, $x \in [0,1]$, $j \in \{0,1,2\}$. Also we set $\mu_{n,s}(x) := \mathcal{K}_n^{(\alpha)}((e_1 - xe_0)^s, x)$, $x \in [0,1]$, the central moment of s order for $\mathcal{K}_n^{(\alpha)}$ operator. We present some identities involving the mentioned test functions and moments.

Lemma 1. *Let $\mathcal{K}_n^{(\alpha)}$ be defined by* (4). *For every $x \in [0,1]$ and $n \in \mathbb{N}$, the following relations hold true*

$$(\mathcal{K}_n^{(\alpha)} e_0)(x) = 1, \tag{5}$$

$$(\mathcal{K}_n^{(\alpha)} e_1)(x) = \beta_n x + (2b_n)^{-1}, \tag{6}$$

$$(\mathcal{K}_n^{(\alpha)} e_2)(x) = \tfrac{\beta_n^2}{\alpha+1}\left(\left(1-\tfrac{1}{n}\right)x^2 + \left(\alpha + \tfrac{1}{n}\right)a_n^{-1}x\right) + \tfrac{\beta_n x}{b_n} + \tfrac{1}{3b_n^2}, \tag{7}$$

$$\mu_{n,1}(x) = (2b_n)^{-1} - (1-\beta_n)x, \tag{8}$$

$$\mu_{n,2}(x) = \mu_{n,1}^2(x) + \beta_n^2 \tfrac{n\alpha+1}{n(\alpha+1)} x(a_n^{-1} - x) + \tfrac{1}{12b_n^2}, \tag{9}$$

where

$$\beta_n := \tfrac{na_n}{b_n}, \quad n \in \mathbb{N}. \tag{10}$$

Proof. By a straightforward calculation we deduce

$$(\mathcal{K}_n^{(\alpha)}e_0)(x) = (S_n^{(\alpha)}e_0)(a_nx), \quad (\mathcal{K}_n^{(\alpha)}e_1)(x) = \left(\tfrac{n}{b_n}S_n^{(\alpha)}e_1 + \tfrac{1}{2b_n}S_n^{(\alpha)}e_0\right)(a_nx),$$

$$(\mathcal{K}_n^{(\alpha)}e_2)(x) = \tfrac{1}{b_n^2}\left(n^2S_n^{(\alpha)}e_2 + nS_n^{(\alpha)}e_1 + \tfrac{1}{3}S_n^{(\alpha)}e_0\right)(a_nx),$$

and taking into account the identities [10, Lemma 4.1]

$$S_n^{(\alpha)}e_j = e_j, \quad j \in \{0,1\}, \quad \text{and} \quad (S_n^{(\alpha)}e_2)(x) = \tfrac{1}{\alpha+1}\left(\tfrac{x(1-x)}{n} + x(x+\alpha)\right),$$

our relations (5), (6), (7) follow. Consequently, the identities (8) and (9) hold also true. □

LEMMA 2. *The second central moment of the operator $\mathcal{K}_n^{(\alpha)}$ verifies*

$$\mu_{n,2}(x) \le \beta_n^2\tfrac{n\alpha+1}{n(\alpha+1)}\varphi_n^2(x) + (1-\beta_n)^2, \quad x \in [0,1], \tag{11}$$

where φ_n is the step-weight function associated to $\mathcal{K}_n^{(\alpha)}$ and defined by

$$\varphi_n(x) = \sqrt{x(a_n^{-1}-x)}, \quad x \in [0,1]. \tag{12}$$

Proof. By using relations (3) and (8), after some algebraic manipulations we get

$$\begin{aligned}\sup_{x\in[0,1]} \mu_{n,1}^2(x) &= \max\left\{\tfrac{1}{4b_n^2}, \left(1-\beta_n-\tfrac{1}{2b_n}\right)^2\right\} = \left(1-\beta_n-\tfrac{1}{2b_n}\right)^2 := c_n \\ &\le (1-\beta_n)^2 - \tfrac{1}{12b_n^2},\end{aligned} \tag{13}$$

and (9) implies the desired result. □

3. APPROXIMATION PROPERTIES OF $\mathcal{K}_n^{(\alpha)}$

THEOREM 1. *Let $\mathcal{K}_n^{(\alpha)}$ be defined by* (4). *For every $f \in C[0,1]$ one has*

$$|(\mathcal{K}_n^{(\alpha)}f)(x) - f(x)| \le 2\omega_f\left(\beta_n\sqrt{\tfrac{n\alpha+1}{n(\alpha+1)}}\varphi_n(x) + 1 - \beta_n\right),$$

where ω_f is the first modulus of continuity of f and β_n, φ_n are defined by (10) *respectively* (12).

Proof. By virtue of the classical results regarding the local rate of convergence, see e.g. the monograph [2, Th. 5.1.2], the identity (5) guarantees

$$|(\mathcal{K}_n^{(\alpha)}f)(x) - f(x)| \le \left(1 + \tfrac{1}{\delta}\sqrt{\mu_{n,2}(x)}\right)\omega_f(\delta), \quad (\forall)\ \delta > 0.$$

We choose $\delta := \sqrt{\mu_{n,2}(x)}$ and knowing that ω_f is a non-decreasing function, with the help of (11) we obtain the claimed result. □

REMARK. B. Lenze [6] introduced the Lipschitz type maximal function $\widetilde{f}_\beta$ of order β, $\beta \in (0,1]$, as follows

$$\widetilde{f}_\beta(x) = \sup_{\substack{x,t\in[0,1] \\ x\ne t}} \frac{|f(x)-f(t)|}{|x-t|^\beta}, \quad x \in [0,1].$$

From the estimate $|f(x)-f(t)| \le \widetilde{f}_\beta(x)\mu_{n,2}^{\beta/2}(x)$ we get

$$\big|(\mathcal{K}_n^{(\alpha)} f)(x) - f(x)\big| \le \widetilde{f}_\beta(x)\Big(\beta_n\sqrt{\tfrac{n\alpha+1}{n(\alpha+1)}}\varphi_n(x) + 1 - \beta_n\Big)^{\beta/2}, \quad x\in[0,1],$$

for every $f \in C[0,1]$. In particular case $a_n = 1$ the relation shows that the order of approximation by $\mathcal{K}_n^{(\alpha)}$ increases near to the endpoint 0 of the interval [0,1]. For $K_n^{(\alpha)}$ operators defined by (2) this type of estimate already appeared in [3, Eq. (1.12)]. □

Theorem 2. *Let $\mathcal{K}_n^{(\alpha)}$ and β_n be defined by* (4) *respectively by* (10). *If*

$$\beta_n \to 1 \quad as \quad n\to\infty, \tag{14}$$

then $\lim_{n\to\infty}\mathcal{K}_n^{(\alpha)} f = f$ uniformly on $[0,1]$ for every $f \in C[0,1]$ as well as $\lim_{n\to\infty}\mathcal{K}_n^{(\alpha)} f = f$ in $L_p[0,1]$ for every $f\in L_p[0,1]$ and $p\ge 1$.

Proof. Under our assumption (14), the relations (5), (6), (7) imply $\lim\limits_{n\to\infty}\mathcal{K}_n^{(\alpha)} e_j = e_j$, $j\in\{0,1,2\}$. By Bohman-Korovkin's theorem and knowing that $C[0,1]$ is dense in every Banach space $L_p[0,1]\subset L_1[0,1]$, $p\ge 1$, the proof is complete. □

Further on, C denotes a constant independent of n and x, which is not necessarily the same at each occurrence. In concordance with the results due to Z. Ditzian and V. Totik [4, pp. 10–11, 24] we set

$$\omega_{\varphi^\lambda}^2(f,t) := \sup_{0<h\le t}\ \sup_{x\pm h\varphi^\lambda\in[0,1]} |\Delta^2_{h\varphi^\lambda} f(x)|, \tag{15}$$

$$Y_\lambda := \{g\in C[0,1] :\ g'\in A.C._{loc},\ \|\varphi^{2\lambda} g''\|<\infty\},$$

$$K_{\varphi^\lambda}(f,t^2) := \inf_{g\in Y_\lambda}\{\|f-g\| + t^2\|\varphi^{2\lambda} g''\|\},$$

$$\bar{Y}_\lambda := \{g\in Y_\lambda :\ \|g''\|<\infty\},$$

$$\bar{K}_{\varphi^\lambda}(f,t^2) := \inf_{g\in\bar{Y}_\lambda}\{\|f-g\| + t^2\|\varphi^{2\lambda} g''\| + t^{4/(2-\lambda)}\|g''\|\}, \tag{16}$$

where $\varphi(x) = \sqrt{x(1-x)}$, $0\le\lambda\le 1$, and $g'\in A.C._{loc}$ means that g is differentiable and g' is absolutely continuous on $[0,1]$.

Regarding the above maps we have the following connections

$$\omega_{\varphi^\lambda}^2(f,t) \sim K_{\varphi^\lambda}(f,t^2) \sim \bar{K}_{\varphi^\lambda}(f,t^2), \quad 0<t\le t_0, \tag{17}$$

established in [4, Th. 2.1.1 & 3.1.2] for the particular case $f\in C[0,1]$. Here $u\sim v$ means that a constant $C>0$ exists with the property $C^{-1}u\le v\le Cu$.

Theorem 3. *Let $\mathcal{K}_n^{(\alpha)}$ be defined by* (4) *such that $a_n = 1$ and $b_n(\alpha+n^{-1}) = \mathcal{O}(1)$, as $n\to\infty$. For $f\in C[0,1]$ and $0\le\lambda\le 1$, one has*

$$\big|(\mathcal{K}_n^{(\alpha)} f)(x) - f(x)\big| \le C\omega_{\varphi^\lambda}^2(f,\Delta_{n,\lambda}(x)) + \omega_f\big(1-\tfrac{2n+1}{2b_n}\big),$$

where $\omega^2_{\varphi^\lambda}$ *is given at* (15),

$$\Delta_{n,\lambda}(x) := b_n^{-1/2}\delta_n^{1-\lambda}(x) \quad \textit{and} \quad \delta_n(x) := \varphi(x) + b_n^{-1/2}. \tag{18}$$

Proof. Since $0 \le x + \mu_{n,1}(x) \le 1$, $x \in [0,1]$, for every $f \in C[0,1]$ we can define

$$(\mathcal{L}_n f)(x) := f(x) - f(x + \mu_{n,1}(x)), \quad (\widetilde{\mathcal{K}}_n^{(\alpha)} f)(x) := (\mathcal{K}_n^{(\alpha)} + \mathcal{L}_n)(f,x). \tag{19}$$

From (5) and (6) we easily obtain $\widetilde{\mathcal{K}}_n^{(\alpha)} e_j = e_j$, $j \in \{0,1\}$. At the same time $\widetilde{\mathcal{K}}_n^{(\alpha)}((e_1 - xe_0)^2, x) = \mu_{n,2}(x) - \mu_{n,1}^2(x)$ and gathering both (7), the additional assumption $b_n(\alpha + n^{-1}) = \mathcal{O}(1)$, as $n \to \infty$, and (18) one obtains

$$\widetilde{\mathcal{K}}_n^{(\alpha)}((e_1 - xe_0)^2, x) \le (\alpha + n^{-1})\varphi^2(x) + b_n^{-2} \le \tfrac{C}{b_n}\delta_n^2(x). \tag{20}$$

On the other hand, for u between t and x we have

$$\tfrac{|t-u|}{\varphi^{2\lambda}(u)} \le \tfrac{|t-x|}{\varphi^{2\lambda}(x)} \quad \text{and} \quad \tfrac{|t-u|}{\delta_n^{2\lambda}(u)} \le \tfrac{|t-x|}{\delta_n^{2\lambda}(x)}. \tag{21}$$

Indeed, if a function $\theta^2 \in \mathbb{R}^{[0,1]}$ is concave then $\theta^{2\lambda}$, $\lambda \in [0,1]$, has the same property and for every $u = (1-\eta)t + \eta x$, $\eta \in [0,1]$, we get $\theta^2(u) \ge (1-\eta)\theta^2(t) + \eta\theta^2(x) \ge \eta\theta^2(x)$. Choosing $\theta^2 = \varphi^2$ respectively $\theta^2 = \delta_n^2$ we obtain (21).

For a given $(x,\lambda) \in [0,1] \times [0,1]$, relations (16) and (17) allow us to choose $g \in \bar{Y}_\lambda$ such that

$$\begin{aligned} \|f - g\| &\le C\omega^2_{\varphi^\lambda}(f, \Delta_{n,\lambda}(x)), \\ \Delta^2_{n,\lambda}(x)\|\varphi^{2\lambda}g''\| &\le C\omega^2_{\varphi^\lambda}(f, \Delta_{n,\lambda}(x)), \\ \Delta^{4/(2-\lambda)}_{n,\lambda}(x)\|g''\| &\le C\omega^2_{\varphi^\lambda}(f, \Delta_{n,\lambda}(x)), \end{aligned}$$

where $\Delta_{n,\lambda}(x)$ is given at (18). Since $\|\widetilde{\mathcal{K}}_n^{(\alpha)}\| \le 3$ we can write

$$\begin{aligned} |(\widetilde{\mathcal{K}}_n^{(\alpha)} f)(x) - f(x)| &\le |\widetilde{\mathcal{K}}_n^{(\alpha)}(f-g,x)| + |(\widetilde{\mathcal{K}}_n^{(\alpha)} g)(x) - g(x)| + |g(x) - f(x)| \\ &\le 4\|f-g\| + |(\widetilde{\mathcal{K}}_n^{(\alpha)} g)(x) - g(x)|. \end{aligned} \tag{22}$$

Also, applying (21) we have

$$\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| \le \|\delta_n^{2\lambda}g''\| \left|\int_x^t \frac{t-u}{\delta_n^{2\lambda}(u)}\mathrm{d}u\right| \le \|\delta_n^{2\lambda}g''\|\delta_n^{-2\lambda}(x)(t-x)^2,$$

$$\begin{aligned} \left|\int_x^{x+\mu_{n,1}(x)} (x + \mu_{n,1}(x) - u)g''(u)\mathrm{d}u\right| &\le \|\delta_n^{2\lambda}g''\| \left|\int_x^{x+\mu_{n,1}(x)} \tfrac{|x+\mu_{n,1}(x)-u|}{\delta_n^{2\lambda}(u)}\mathrm{d}u\right| \\ &\le \|\delta_n^{2\lambda}g''\| \frac{|\mu_{n,1}(x)|}{\delta_n^{2\lambda}(x)} \left|\int_x^{x+\mu_{n,1}(x)} \mathrm{d}u\right| \\ &= \|\delta_n^{2\lambda}g''\|\delta_n^{-2\lambda}\mu^2_{n,1}(x). \end{aligned}$$

Using (19), the above two inequalities as well as (20) we have

$$\begin{aligned}\big|(\widetilde{\mathcal{K}}_n^{(\alpha)}g)(x)-g(x)\big| &= \left|\widetilde{\mathcal{K}}_n^{(\alpha)}\Big(\int_{xe_0}^{e_1}(e_1-u)g''(u)\mathrm{d}u,x\Big)\right| \\ &\le \left|\mathcal{K}_n^{(\alpha)}\Big(\int_{xe_0}^{e_1}(e_1-u)g''(u)\mathrm{d}u,x\Big)\right| \\ &\quad + \left|\int_x^{x+\mu_{n,1}(x)}(x+\mu_{n,1}(x)-u)g''(u)\mathrm{d}u\right| \\ &\le \|\delta_n^{2\lambda}g''\|\delta_n^{-2\lambda}(x)\mu_{n,2}(x)+\|\delta_n^{2\lambda}g''\|\delta_n^{-2\lambda}(x)\mu_{n,1}^2(x) \\ &\le \|\delta_n^{2\lambda}g''\|\delta_n^{-2\lambda}(x)\{2\mu_{n,1}^2(x)+\widetilde{\mathcal{K}}_n^{(\alpha)}((e_1-xe_0)^2,x)\} \\ &\le C\Delta_{n,\lambda}^2(x)\|\delta_n^{2\lambda}g''\|.\end{aligned}$$

In the same manner we establish

$$\big|(\widetilde{\mathcal{K}}_n^{(\alpha)}g)(x)-g(x)\big| \le C\tfrac{\delta_n^2(x)}{b_n}\varphi^{-2\lambda}(x)\|\varphi^{2\lambda}g''\|.$$

We split $I:=[0,1]$ in two parts: E_n and $I\setminus E_n$ where $E_n:=\Big[\frac{A}{n},1-\frac{A}{n}\Big]$, A being a fixed positive number.

For $x\in E_n$ we have $\delta_n(x)\sim\varphi(x)$. By using (22) and (17) we get

$$\big|(\widetilde{\mathcal{K}}_n^{(\alpha)}f)(x)-f(x)\big| \le 4\|f-g\|+\tfrac{C}{b_n}\delta_n^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\| \le C\omega_{\varphi^\lambda}^2(f,\Delta_{n,\lambda}(x)).$$

For $x\in I\setminus E_n$ we have $\delta_n(x)\sim b_n^{-1/2}$, therefore

$$(\delta_n^{2(1-\lambda)}(x)/b_n^{\lambda+1})\sim(\delta_n^{4(1-\lambda)/(2-\lambda)}(x)/b_n^{2/(2-\lambda)}).$$

Based on the previous increases, we get

$$\begin{aligned}\big|(\widetilde{\mathcal{K}}_n^{(\alpha)}f)(x)-f(x)\big| &\le 4\|f-g\|+\tfrac{C}{b_n}\delta_n^{2(1-\lambda)}(x)\Big\{\|\varphi^{2\lambda}g''\|+\tfrac{1}{b_n^\lambda}\|g''\|\Big\} \\ &\le C\Big\{\|f-g\|+\Delta_{n,\lambda}^2(x)\|\varphi^{2\lambda}g''\|+\Delta_{n,\lambda}^{4/(2-\lambda)}(x)\|g''\|\Big\} \\ &\le C\omega_{\varphi^\lambda}^2(f,\Delta_{n,\lambda}(x)).\end{aligned}$$

Consequently, for every $f\in C[0,1]$ and $x\in[0,1]$ we have

$$\begin{aligned}\big|(\mathcal{K}^{(\alpha)}f)(x)-f(x)\big| &\le \big|(\widetilde{\mathcal{K}}_n^{(\alpha)}f)(x)-f(x)\big|+\big|(\mathcal{L}_nf)(x)\big| \\ &\le C\omega_{\varphi^\lambda}^2(f,\Delta_{n,\lambda}(x))+\omega_f(|\mu_{n,1}(x)|)\end{aligned}$$

and (13) finished the proof. □

We notice that the above theorem generalizes a result which was recently obtained for the Bernstein-Kantorovich operators $\mathcal{K}_n^{(0,1,n+1)}=K_n^{(0)}$, see [5, Th. 3].

We end this section going to study the degree of approximation for f belonging to $L_p[0,1]$, $p\ge1$, by using the integral modulus of smoothness of high order $\omega_r(f,t)_p:=\sup_{0<|h|\le t}\|(T_h-I)^rf\|_p$, T_h being the translation operator.

According to (13) it is clear that $\|\mu_{n,1}\|_p \leq \sqrt{c_n}$. Further on, by using Minkowski inequality from (11) we deduce

$$\|\mu_{n,2}\|_p \leq \beta_n\sqrt{\alpha+n^{-1}}\left(\int_0^1 \varphi_n^p(x)\mathrm{d}x\right)^{1/p} + 1 - \beta_n := \gamma_{n,p}. \tag{23}$$

THEOREM 4. *Let $\mathcal{K}_n^{(\alpha)}$ be defined by* (4). *For every $f \in L_p[0,1]$, $p \geq 1$, the following inequalities*

$$\|\mathcal{K}_n^{(\alpha)}f - f\|_p \leq C_{p,r}\Big(\gamma_{n,p}\|f\|_p + \omega_r(f, 2r\gamma_{n,p}^{1/r})_p\Big), \quad n \geq n_0, \tag{24}$$

hold, where $C_{p,r}$ is a constant independent of f and n, $\gamma_{n,p}$ is given at (23) *and $r \geq 3$ is an integer.*

Proof. Since $\gamma_{n,p} = o(1)$, as $n \to \infty$, for an integer $r \geq 3$ a rank n_0 exists such that $2r\gamma_{n,p}^{1/r} \leq 1$ for every $n \geq n_0$. The proof of (24) follows the same steps like those established in [1, Th. 1], so we overlook it. □

4. THE ITERATES OF $S_n^{(\alpha)}$ VIA CONTRACTION PRINCIPLE

In [7] the iterates ${}^mS_n^{(\alpha)}$, $m \geq 0$, of Stancu operators have been introduced and investigated. We recall

$${}^0S_n^{(\alpha)} = 1, \quad {}^1S_n^{(\alpha)} = S_n^{(\alpha)}, \quad {}^mS_n^{(\alpha)} = S_n^{(\alpha)}\big({}^{m-1}S_n^{(\alpha)}\big), \quad m > 1.$$

The authors proved the following limiting relation

$$\lim_{m\to\infty} {}^mS_n^{(\alpha)}(f,x) = f(0) + (f(1) - f(0))x, \tag{25}$$

uniformly on $[0,1]$ for any $\alpha \geq 0$.

The aim of this section is to give a new proof of (25). Our approach is motivated by the results due to I.A. Rus [9].

At first we define

$$X_{\alpha,\beta} := \{f \in C[0,1] : \ f(0) = \alpha, \ f(1) = \beta\},$$

for every real parameter α and β. It is easy to observe that $X_{\alpha,\beta}$ is a closed subset of $C[0,1]$, it is an invariant subset of $S_n^{(\alpha)}$ for all $n \in \mathbb{N}$, and $X_{\alpha,\beta}$, $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$, form a partition of $C[0,1]$.

The next step we prove that $S_n^{(\alpha)}|_{X_{\alpha,\beta}} : X_{\alpha,\beta} \to X_{\alpha,\beta}$ is a contraction for every $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ and $n \in \mathbb{N}$. Considering $f, g \in X_{\alpha,\beta}$ and knowing that

$S_n^{(\alpha)}h$ interpolates the function h in 0 and 1, from (1) we can write

$$\begin{aligned} \left|(S_n^{(\alpha)}f)(x)-(S_n^{(\alpha)}g)(x)\right| &= \left|\sum_{k=1}^{n-1} w_{n,k}^{(\alpha)}(x)(f-g)\left(\tfrac{k}{n}\right)\right| \\ &\le \left(1-w_{n,0}^{(\alpha)}(x)-w_{n,n}^{(\alpha)}(x)\right)\|f-g\| \\ &\le \left(1-\tfrac{(1-x)^n+x^n}{1^{[n,-\alpha]}}\right)\|f-g\| \\ &\le \left(1-\tfrac{2^{1-n}}{1^{[n,-\alpha]}}\right)\|f-g\|. \end{aligned}$$

Since $S_n^{(\alpha)}$ has the exactness degree 1, obviously $\alpha e_0+(\beta-\alpha)e_1$ is a fixed point of $S_n^{(\alpha)}|_{X_{\alpha,\beta}}$.

If $f\in C[0,1]$ then $f\in X_{f(0),f(1)}$ and from the contraction principle we obtain (25). □

Remark. According to [9, Th. 1'] the above used trend allow us to state that Stancu operator $S_n^{(\alpha)}$ is a weakly Picard operator. □

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