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ON A THIRD ORDER ITERATIVE METHOD FOR SOLVING POLYNOMIAL OPERATOR EQUATIONS*

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Abstract. We present a semilocal convergence result for a Newton-type method applied to a polynomial operator equation of degree 2. The method consists in fact in evaluating the Jacobian at every two steps, and it has the r-convergence order at least 3.

We apply the method in order to approximate the eigenpairs of matrices. We perform some numerical examples on some test matrices and compare the method to the Chebyshev method. The norming function we have proposed in a previous paper shows a better convergence of the iterates than the classical norming function for both the methods.

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1. INTRODUCTION

Let $F : X \to X$ be a nonlinear mapping, where $(X, \|\cdot\|)$ is a Banach space, and consider the equation

$$F(x) = 0. \tag{1}$$

We shall assume that F is a polynomial operator of degree 2, i.e., it is indefinitely differentiable on X, with $F^{(i)}(x) = \theta_i$, for all $x \in X$ and $i \geq 3$, where θ_i is the i-linear null operator.

Besides (1) we shall also consider another equation, equivalent with it

$$x - G(x) = 0, \tag{2}$$

where $G : X \to X$. More exactly, we shall assume that the solutions of (1) coincide with the solutions of (2) and viceversa.

In [15] it was shown that the following iterations

$$x_{k+1} = G(x_k) - F'(x_k)^{-1} F(G(x_k)), \quad k = 0, 1, \ldots, \; x_0 \in X \tag{3}$$

have the convergence order with one order higher than the convergence order of the iterates $x_{k+1} = G(x_k)$.

Obviously, if we take as G the Newton operator, i.e.,

$$G(x) = x - F'(x)^{-1} F(x), \tag{4}$$

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then we obtain a method with the convergence order at least 3.

In the present paper we shall study the convergence of the iterations (3), with G given by (4), i.e.,

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k) - F'(x_k)^{-1} F\big(x_k - F'(x_k)^{-1} F(x_k)\big), \tag{5}$$

in order to solve the polynomial equation (1).

By (5), for some known approximation x_k, the next approximation x_{k+1} may be determined as

1. Solve $F'(x_k)\, s = -F(x_k)$
   Set $u = x_k + s$
2. Solve $F'(x_k)\, t = -F(u)$
   Set $x_{k+1} = u + t$.

We notice that at each iteration step we need to solve two linear equations, but for the same linear operator, $F'(x_k)$. The iterations may be viewed as being given by the Newton method in which the Jacobian is evaluated at every two steps.

A possible advantage of the above method over the Chebyshev method

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k) - \tfrac{1}{2} F'(x_k)^{-1} F''(x_k) \left(F'(x_k)^{-1} F(x_k)\right)^2,$$

which has the same convergence order, is that it does not require the second derivative of F, which may have a complicate form.

The study of such methods for second degree polynomial equations is important, since such equations often arise in practice. We mention the eigenvalue problem (see, e.g., [21], [20]), some integral equations (see, e.g., [2], [8]), etc.

We shall apply this study to the approximation of the eigenpairs of the matrices, and we shall consider some numerical examples for some test matrices. We shall also compare the method (5) to the Chebyshev method.

2. A SEMILOCAL CONVERGENCE RESULT

Since F is a second degree polynomial, one can easily show that

$$F(x) = F(y) + F'(y)(x-y) + \tfrac{1}{2} F''(y)(x-y)^2, \quad \forall x, y \in X. \tag{6}$$

Assuming that $F'(x)^{-1}$ exists, denote by $\varphi(x)$ the following expression:

$$\varphi(x) = -F'(x)^{-1} F(x) - F'(x)^{-1} F\big(x - F'(x)^{-1} F(x)\big). \tag{7}$$

Taking into account (6), we get

$$F\big(x - F'(x)^{-1} F(x)\big) = \tfrac{1}{2} F''(x) \left(F'(x)^{-1} F(x)\right)^2. \tag{8}$$

For the given norm $\|\cdot\|$, by (6)–(8) it follows that

$$\begin{aligned} &\left\| F(x) + F'(x)\varphi(x) + \tfrac{1}{2} F''(x)\varphi^2(x) \right\| \leq \\ &\leq \tfrac{1}{2} \|F''(x)\|^2 \|F'(x)^{-1}\|^4 \|F(x)\|^3 + \tfrac{1}{8} \left\|F''(x)\right\|^3 \|F'(x)^{-1}\|^6 \|F(x)\|^4 . \end{aligned} \tag{9}$$

Analogously, by (7) and (8), we get for all $x \in X$

$$\|\varphi(x)\| \le \|F'(x)^{-1}\| \cdot \|F(x)\| + \tfrac{1}{2}\|F''(x)\| \|F'(x)^{-1}\|^3 \|F(x)\|^2 . \tag{10}$$

Since the bilinear operator $F''(x)$ does not depend on x, denote $K = \|F''(x)\|$.

Let $x_0 \in X$, $r > 0$, denote $\beta_0 = \|F'(x_0)^{-1}\|$, and assume that $\beta_0 K r < 1$. Using the Banach lemma, it easily follows that for all $x \in \bar{B}_r(x_0) = \{x \in X : \|x - x_0\| \le r\}$, there exists $F'(x)^{-1}$, and

$$\|F'(x)^{-1}\| \le \frac{\beta_0}{1 - \beta_0 K r} \overset{not}{=} \beta. \tag{11}$$

With the above notations, by (9) and (10) it follows for any $x \in \bar{B}_r(x_0)$ that

$$\begin{aligned} &\left\|F(x) + F'(x)\varphi(x) + \tfrac{1}{2}F''(x)\varphi^2(x)\right\| \le \\ &\le \tfrac{1}{2}\beta^4 K^2 \left(1 + \tfrac{1}{4}\beta^2 K \|F(x)\|\right) \|F(x)\|^3 \end{aligned} \tag{12}$$

$$\|\varphi(x)\| \le \beta \left(1 + \tfrac{1}{2}\beta^2 K \|F(x)\|\right) \|F(x)\| . \tag{13}$$

Now take

$$\begin{aligned} a_0 &= \tfrac{1}{2}\beta^4 K^2 \left(1 + \tfrac{1}{4}\beta^2 K \|F(x_0)\|\right), \\ b_0 &= \beta \left(1 + \tfrac{1}{2}\beta^2 K \|F(x_0)\|\right). \end{aligned} \tag{14}$$

From (12), taking into account (5) and (6), we get

$$\begin{aligned} \|F(x_1)\| &= \left\|F(x_0) + F'(x_0)(x_1 - x_0) + \tfrac{1}{2}F''(x_0)(x_1 - x_0)^2\right\| \\ &= \left\|F(x_0) + F'(x_0)\varphi(x_0) + \tfrac{1}{2}F''(x_0)\varphi^2(x_0)\right\| \\ &\le a_0 \|F(x_0)\|^3 . \end{aligned} \tag{15}$$

Relations (5) and (13) imply

$$\|x_1 - x_0\| \le b_0 \|F(x_0)\| . \tag{16}$$

We obtain the following result regarding the convergence of (5).

Theorem 1. *If the mapping F, the initial approximation $x_0 \in X$, and the real numbers β_0, K, r satisfy:*

i. *F is a second degree polynomial;*
ii. *$\beta_0 K r < 1$, assuming that $F'(x_0)^{-1}$ exists and $\|F'(x_0)^{-1}\| = \beta_0$;*
iii. *$q = \sqrt{a_0}\, \|F(x_0)\| < 1$;*

iv. $\frac{b_0 q}{\sqrt{a_0}(1-q^2)} \leq r$,

then the following statements are true:

j. *the sequence* $(x_k)_{k\geq 0}$ *given by* (5) *is well defined;*

jj. *equation* (1) *has (at least) a solution* $x^* \in \bar{B}_r(x_0)$;

jjj. *one has the estimates*

$$\|x^* - x_k\| \leq \frac{b_0 q^{3^k}}{\sqrt{a_0}(1-q^2)}, \tag{17}$$

$$\|x_{k+1} - x_k\| \leq b_0 \|F(x_k)\|, \quad k = 0, 1, \ldots \tag{18}$$

Proof. Assumptions iii. and (6) imply that $x_1 \in \bar{B}_r(x_0)$.

Relation (15) may also be written as

$$\|F(x_1)\| \leq \frac{1}{\sqrt{a_0}} q^3. \tag{19}$$

which, by ii., leads to $\|F(x_1)\| < \|F(x_0)\|$. Denoting

$$a_1 = \tfrac{1}{2}\beta^4 K^2 \left(1 + \tfrac{1}{4}\beta^2 K \|F(x_1)\|\right), \quad b_1 = \beta \left(1 + \tfrac{1}{2}\beta^2 K \|F(x_1)\|\right)$$

then, obviously, $a_1 < a_0$ and $b_1 < b_0$.

Assume now the following relations:

a) $x_i \in \bar{B}_r(x_0), \quad i = 1, \ldots, k$;

b) $\|F(x_i)\| \leq \frac{1}{\sqrt{a_0}} q^{3^i}, \quad i = 1, \ldots, k$;

c) $a_i < a_{i-1}$ and $b_i < b_{i-1}$, $i = 1, \ldots, k$.

We shall prove that

$$x_{k+1} \in \bar{B}_r(x_0), \quad \|F(x_{k+1})\| \leq \frac{1}{\sqrt{a_0}} q^{3^{k+1}}, \quad a_{k+1} < a_k, \quad b_{k+1} < b_k, \tag{20}$$

where

$$a_i = \tfrac{1}{2}\beta^4 K^2 \left(1 + \tfrac{1}{4}\beta^2 K \|F(x_i)\|\right), \quad b_i = \beta \left(1 + \tfrac{1}{2}\beta^2 K \|F(x_i)\|\right),$$

$i = 1, \ldots, k+1$.

From (5) and (7), we deduce

$$\|x_{k+1} - x_k\| \leq \|\varphi(x_k)\| \leq b_k \|F(x_k)\| \leq \frac{b_k}{\sqrt{a_0}} q^{3^k}.$$

But $b_k < b_0$ and hence

$$\|x_{k+1} - x_k\| \leq \frac{b_0}{\sqrt{a_0}} q^{3^k}.$$

This implies

$$\|x_{k+1} - x_0\| \leq \sum_{i=0}^{k} \|x_{i+1} - x_i\| \leq \frac{b_0}{\sqrt{a_0}} \sum_{i=0}^{k} q^{3^i} \leq \frac{b_0 q}{\sqrt{a_0}(1-q^2)} \leq r,$$

i.e., $x_{k+1} \in \bar{B}_r(x_0)$.

From (5) and (12) we obtain $\|F(x_{k+1})\| \leq a_k \|F(x_k)\|^3$, and taking into account that $\sqrt{a_k}\,\|F(x_k)\| < 1$, we get $\|F(x_{k+1})\| < \|F(x_k)\|$, i.e., $a_{k+1} < a_k$ and $b_{k+1} < b_k$.

Hence

$$\|F(x_{k+1})\| \leq \tfrac{1}{\sqrt{a_0}} q^{3^{k+1}}.$$

Now we show that the sequence $(x_k)_{k\geq 0}$ is fundamental. From the above relations we have

$$\|x_{k+m} - x_k\| \leq \sum_{i=k}^{m-1} \|x_{i+1} - x_i\| \leq \tfrac{b_0}{\sqrt{a_0}} \sum_{i=k}^{m-1} q^{3^i} \leq \tfrac{b_0 q^{3^k}}{\sqrt{a_0}(1-q^2)}, \tag{21}$$

for all $m, k \in \mathbb{N}$. Since $q < 1$, we get that the sequence is Cauchy. Denote $x^* = \lim_{k\to\infty} x_k$.

By (21), for $m \to \infty$ we get

$$\|x^* - x_k\| \leq \tfrac{b_0 q^{3^k}}{\sqrt{a_0}(1-q^2)}, \quad k = 0, 1, \ldots \ .$$

The continuity of F implies that $F(x^*) = 0$. Obviously, $x^* \in \bar{B}_r(x_0)$. $\square$

3. APPLICATION AND NUMERICAL EXAMPLES

We shall study this method when applied to approximate the eigenpairs of matrices.

Denote $V = \mathbb{K}^n$ and let $A \in \mathbb{K}^{n\times n}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For computing the eigenpairs of A one may consider a norming function $G : V \to \mathbb{K}$ with $G(0) \neq 1$. The eigenvalues $\lambda \in \mathbb{K}$ and eigenvectors $v \in V$ of A are the solutions of the nonlinear system

$$F(x) = \begin{pmatrix} Av - \lambda v \\ G(v) - 1 \end{pmatrix} = 0,$$

where $x = \binom{v}{\lambda} \in V \times \mathbb{K} = \mathbb{K}^{n+1}$, $\left(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right) = v$ and $x^{(n+1)} = \lambda$. The first n components of F, F_i, $i = 1, \ldots, n$, are given by

$$F_i(x) = a_{i1}x^{(1)} + \ldots + a_{i,i-1}x^{(i-1)} + \left(a_{ii} - x^{(n+1)}\right)x^{(i)} + a_{i,i+1}x^{(i+1)} + \ldots + a_{in}x^{(n)}.$$

The standard choice for G is

$$G(v) = \alpha \|v\|_2,$$

with $\alpha = \frac{1}{2}$. We have proposed in [4] (see also [7]), the choice $\alpha = \frac{1}{2n}$, which has shown a better behavior for the iterates than the standard choice.

In both cases we can write

$$F_{n+1}(x) = \alpha\Big(\left(x^{(1)}\right)^2 + \ldots + \left(x^{(n)}\right)^2\Big) - 1.$$

The first and the second order derivatives of F are given by

$$F'(x)\,h = \begin{pmatrix} a_{11}-x^{(n+1)} & a_{12} & \dots & a_{1n} & -x^{(1)} \\ a_{21} & a_{22}-x^{(n+1)} & \dots & a_{2n} & -x^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-x^{(n+1)} & -x^{(n)} \\ 2\alpha x^{(1)} & 2\alpha x^{(2)} & \dots & 2\alpha x^{(n)} & 0 \end{pmatrix} \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ \vdots \\ h^{(n)} \\ h^{(n+1)} \end{pmatrix},$$

and

$$F''(x)\,hk = \begin{pmatrix} -k^{(n+1)} & 0 & \dots & 0 & -k^{(1)} \\ 0 & -k^{(n+1)} & \dots & 0 & -k^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -k^{(n+1)} & -k^{(n)} \\ 2\alpha k^{(1)} & 2\alpha k^{(2)} & \dots & 2\alpha k^{(n)} & 0 \end{pmatrix} \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ \vdots \\ h^{(n)} \\ h^{(n+1)} \end{pmatrix},$$

where $x = \left(x^{(i)}\right)_{i=\overline{1,n+1}}$, $h = \left(h^{(i)}\right)_{i=\overline{1,n+1}}$, $k = \left(k^{(i)}\right)_{i=\overline{1,n+1}} \in \mathbb{K}^{n+1}$.

We shall consider two test matrices from the Harwell Boeing collection[1] in order to study the behavior of the method (5) and of the Chebyshev method for approximating the eigenpairs. The programs were written in Matlab. As in [21], we used the Matlab operator '\' for solving the linear systems.

Fidap002 matrix. This real symmetric matrix of dimension $n = 441$ arises from finite element modeling. Its eigenvalues are all simple and range from $-7 \cdot 10^8$ to $3 \cdot 10^6$. As in [21], we have chosen to study the smallest eigenvalue, which is well separated. The initial approximations were taken $\lambda_0 = \lambda^* + 10^2 = -6.999\,6 \cdot 10^8 + 100$, and for the initial vector v_0 we perturbed the solution v^* (computed by Matlab and then properly scaled to fulfill the norming equation) with random vectors having the components uniformly distributed on $(-\varepsilon,\varepsilon)$, $\varepsilon = 0.5$. The following results are typical for the runs made (we have considered a common perturbation vector); Table 1 contains the norms of the vectors $F(x_k)$. For the choice I, we took $a = \frac{1}{2}$ in G, while for the choice II, $a = \frac{1}{2n}$.

It is interesting to note that the norm of F (even at the computed solution) does not decrease below 10^{-8}.

Sherman1 matrix. This matrix arises from oil reservoir simulation. It is real, unsymmetric, of dimension 1000 and all its eigenvalues are real. We have chosen to study the smallest eigenvalue $\lambda^* = -5.0449$, which is not well separated (the closest eigenvalue is -4.9376). The initial approximation was taken $\lambda_0 = \lambda^* - 0.002$ and for the initial vector v_0 we considered $\varepsilon = 0.01$.

[1] These matrices are available from MatrixMarket at the following address: `http://math.nist.gov/MatrixMarket/`.

Table 1. The Fidap002 matrix.

	Choice I		Choice II	
k	Method (5)	Chebyshev method	Method (5)	Chebyshev method
0	$1.0003 \cdot 10^{+2}$	$1.7467 \cdot 10^{+9}$	$2.5107 \cdot 10^{+0}$	$1.7467 \cdot 10^{+9}$
1	$1.0078 \cdot 10^{+1}$	$8.7335 \cdot 10^{+8}$	$1.2553 \cdot 10^{+0}$	$8.7335 \cdot 10^{+8}$
2	$3.4853 \cdot 10^{+1}$	$1.6462 \cdot 10^{+2}$	$5.2877 \cdot 10^{-2}$	$3.5641 \cdot 10^{-3}$
3	$2.1368 \cdot 10^{+0}$	$2.3373 \cdot 10^{+1}$	$6.1804 \cdot 10^{-5}$	$4.9048 \cdot 10^{-6}$
4	$4.4761 \cdot 10^{-1}$	$6.3521 \cdot 10^{-1}$	$5.9858 \cdot 10^{-7}$	$4.4309 \cdot 10^{-6}$
5	$6.5214 \cdot 10^{-3}$	$9.2238 \cdot 10^{-3}$		
6	$1.5617 \cdot 10^{-5}$	$2.2961 \cdot 10^{-5}$		
7	$5.9605 \cdot 10^{-7}$	$8.5647 \cdot 10^{-8}$		

The following results are typical for the runs made (we have considered again a same random perturbation vector for the four initial approximations).

Table 2. Pores1 matrix.

	Choice I		Choice II	
k	Method (5)	Chebyshev method	Method (5)	Chebyshev method
0	$7.7177 \cdot 10^{-01}$	$7.7177 \cdot 10^{-01}$	$7.7638 \cdot 10^{-01}$	$7.7638 \cdot 10^{-01}$
1	$6.9242 \cdot 10^{-05}$	$3.2237 \cdot 10^{-02}$	$1.2188 \cdot 10^{-06}$	$3.2248 \cdot 10^{-05}$
2	$2.3593 \cdot 10^{-13}$	$4.9948 \cdot 10^{-04}$	$2.7541 \cdot 10^{-14}$	$5.1973 \cdot 10^{-10}$
3	$7.1171 \cdot 10^{-16}$	$1.2466 \cdot 10^{-07}$		$4.0207 \cdot 10^{-14}$
4		$7.8143 \cdot 10^{-15}$		
5		$9.1655 \cdot 10^{-16}$		

For this particular matrix and eigenvalue, the Chebyshev method has shown a greater sensitivity to the size of the perturbations than method (5). Increasing ε leads to the loss of the convergence of the Chebyshev iterates, while method (5) still converge.

Though for the Sherman1 matrix method (5) displayed a better behavior than the Chebyshev method, some extensive tests must be performed before affirming that the first method is superior. In any case, the choice II has shown again that is more advantageous to use.

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