PHELPS TYPE DUALITY RESULTS IN BEST APPROXIMATION

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Abstract. The aim of the present paper is to show that many Phelps type duality result, relating the extension properties of various classes of functions (continuous, linear continuous, bounded bilinear, Hölder-Lipschitz) with the approximation properties of some annihilating spaces, can be derived in a unitary and simple way from a formula for the distance to the kernel of a linear operator, extending the well-known distance formula to hyperplanes in normed spaces. The case of spaces $c_0$ and $l^\infty$ is treated in details.


Keywords. best approximation, Hahn-Banach extension, $M$-ideals.

THE DISTANCE FORMULA

Let $X$ be a normed space (over $\mathbb{R}$ or $\mathbb{C}$) and $Y$ a closed subset of $X$. For $x \in X$ put

$$d(x,Y) = \inf \{ \|x - y\| : y \in Y \},$$

$$P_Y(x) = \{ y \in Y : \|x - y\| = d(x,Y) \}.$$

The quantity $d(x,Y)$ is the distance from $x$ to $Y$ and the elements in $P_Y(x)$ are called nearest points (or elements of best approximation) for $x$ in $Y$. The set-valued map $P_Y$ is called the metric projection. The subspace $Y$ is called proximinal if $P_Y(x) \neq \emptyset$, for every $x \in X$, Chebyshevian if $P_Y(x)$ is a singleton for every $x \in X$, and antiproximinal if $P_Y(x) = \emptyset$, for every $x \in X \setminus Y$ (observe that $P_Y(y) = \{ y \}$, for $y \in Y$).

Denote by $X^*$ the conjugate space of $X$ and let

$$Y^\perp = \{ x^* \in X^* : x^*|_Y = 0 \}$$

be the annihilator of $Y$ in $X^*$. In the seminal paper [38] R. R. Phelps initiated the study of the relations between the extension properties of the space $Y$ and the approximation properties of its annihilator $Y^\perp$. Namely, $Y^\perp$ is Chebyshevian if and only if every functional $y^* \in Y^*$ has a unique norm-preserving extension $x^* \in X^*$. It is known that, by Hahn-Banach theorem, every $y^* \in Y^*$ has at least one norm-preserving extension. Since then, there have been found a lot of situations in which similar duality results hold, corresponding to various extension results – Helly extension theorem for linear functionals, Tietze’s extension theorem for continuous functions, McShane’s extension theorem for

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Lipschitz functions, Nachbin’s extension theorem for continuous linear operators etc.

The aim of the present paper is to show that all these results follow immediately from a formula for the distance to the kernel of a continuous linear operator, inspired by the well-known distance formula to hyperplanes in normed spaces.

For a continuous linear operator \( A : X_1 \to X_2 \), between two normed spaces \( X_1, X_2 \), let

\[
Z = \{ x \in X_1 : Ax = 0 \}
\]

be its kernel. Also, for \( x \in X_1 \), put

\[
E(x) = \{ y \in X_1 : Ay = Ax \text{ and } \|y\| = \|Ax\| \|A\| \}
\]

1. \( d(x, Z) \geq \|Ax\| \|A\| \)
2. We have

\[
d(x, Z) = \frac{\|Ax\|}{\|A\|}
\]

if and only if there exists a sequence \((z_n)\) in \(Z\) such that

\[
\|x - z_n\| \to \|Ax\| \|A\|.
\]

3 (a). If (4) holds then

\[
P_Z(x) = x - E(x)
\]

(with the convention \(x - \emptyset = \emptyset\)).

3 (b). If there is \(z_0 \in Z\) such that

\[
\|x - z_0\| = \frac{\|Ax\|}{\|A\|}
\]

then \(z_0 \in P_Z(x)\) and the formulae (4) and (6) hold.

**Proof.** 1. For every \(z \in Z\), we have

\[
\|Ax\| = \|A(x - z)\| \leq \|A\| \|x - z\|
\]

showing that (5) holds.

2. Let \((z_n) \subset Z\) verifying (5). Then

\[
d(x, Z) \leq \|x - z_n\|, \forall n \in \mathbb{N},
\]

and, letting \(n \to \infty\), one obtains

\[
d(x, Z) \leq \frac{\|Ax\|}{\|A\|}
\]

which, combined with the point 1 of the theorem, yields (4).

Conversely, if the equality (1) holds and \((z_n)\) is a sequence in \(Z\) such that \(\|x - z_n\| \to d(x, Z)\), then the sequence \((z_n)\) verifies (5).
3. (a) Follows from the equivalences:
\[ z \in P_Z(x) \iff z \in Z \quad \text{and} \quad \|x - z\| = d(x, Z) = \frac{\|Ax\|}{\|A\|} \]
\[ \iff x - z \in E(x) \iff z \in x - E(x). \]

3. (b) Observe that the equality (7) implies that (5) holds with \( z_n = z_0, n = 1, 2, \ldots \), so that, by the point 2 of the theorem, (4) and (6) hold too. Since
\[ \|x - z_0\| = \frac{\|Ax\|}{\|A\|} = d(x, Z) \]

it follows that \( z_0 \in P_Z(x) \).

**Remark.** If \( y \in X_1 \) is fixed and \( W := y + Z \) then
\[ d(x, W) = d(x - y, Z) = \frac{\|Ax - Ay\|}{\|A\|}. \]

**Examples**

1. *The distance from a point to a hyperplane*

Let \( X \) be a normed space, \( x^* \in X^* \), \( x^* \neq 0 \), and \( Z = \ker x^* \). Take
\[ A := x^* : X \to K \]
\((K = \mathbb{R} \text{ or } K = \mathbb{C})\).

Let \( x \in X \). First show that condition (5) is always fulfilled. Indeed, if \( u_n \in X, \|u_n\| = 1, n \in \mathbb{N} \), are such that \( |x^*(u_n)| \to ||x^*|| \), then
\[ z_n := x - \frac{x^*(x)}{x^*(u_n)}u_n \in Z \quad \text{and} \quad \|x - z_n\| = \frac{|x^*(x)|}{|x^*(u_n)|} \to \frac{|x^*(x)|}{\|x^*\|}. \]

Therefore
\[ d(x, Z) = \frac{|x^*(x)|}{\|x^*\|} \]

for every \( x \in X \).

Observe now that condition (7) holds if and only if \( x^* \) supports the closed unit ball \( B \) of \( X \). Indeed, if \( u_0 \in X, \|u_0\| = 1 \), is such that \( |x^*(u_0)| = ||x^*|| \), then \( z_0 := x - (x^*(x)/x^*(u_0))u_0 \) is in \( Z \) and
\[ \|x - z_0\| = \frac{|x^*(x)|}{|x^*(u_0)|} = \frac{|x^*(x)|}{\|x^*\|}. \]

Conversely, if, for some \( x_0 \in X \), there is an element \( z_0 \in Z \) such that
\[ \|x - z_0\| = \frac{|x^*(x)|}{\|x^*\|}, \]
then
\[ |x^*(x_0 - z_0)| = |x^*(x)| = ||x^*||\|x_0 - z_0\|, \]

showing that \( x^* \) attains its norm on the element \( u_0 = (x_0 - z_0)/\|x_0 - z_0\| \) of \( B \).

In fact we have shown that if \( x^* \) supports the unit ball \( B \) of \( X \) then (7) holds for every \( x \in X \), and if (7) holds for a single element \( x_0 \in X \setminus Z \) then
$x^*$ supports the unit ball $B$ and, therefore, (7) holds for every $x \in X$. It follows
that the subspace $Z = \ker x^*$ is proximinal if $x^*$ supports the unit ball of $X$, and antiproximinal if not.

If $h_0 \in X$ and $H = h_0 + Z = \{ x \in X : x^*(x) = a \}$ where $a = x^*(h_0)$, is a
closed hyperplane parallel to $Z$ then, by (8),
\[
d(x, H) = \frac{|x^*(x) - a|}{\|x^*\|}
\]
a well-known formula.

2. Restriction operators

Let $E$ be a normed space and $S, T$ nonvoid sets with $S \subset T$. Consider
two normed spaces $X_1 = X_1(T, E)$ and $X_2 = X_2(S, E)$ of mappings from $T$
(respectively $S$) to $E$, the vector operations being defined pointwise. Suppose
that there are verified the following conditions
\[
x|_S \in X_2 \quad \text{and} \quad \|x|_S\| \leq \|x\|
\]
for every $x \in X_1$. For $y \in X_2$ denote by
\[
E(y) = \{ x \in X_1 : x|_S = y \quad \text{and} \quad \|x\| = \|y\| \}
\]
the (possibly empty) set of norm-preserving extensions of $y$ in $X_1$. One says
that the space $X_2$ has the extension property with respect to $X_1$ if $E(y) \neq \emptyset$, for every $y \in X_2$. Let $A : X_1 \to X_2$ be the restriction operator defined by
\[
Ax = x|_S, \quad x \in X_1.
\]
By (9), $A$ is well defined, linear, continuous, and
\[
\|A\| \leq 1.
\]
Put
\[
S^\perp = \{ x \in X_1 : x|_S = 0 \} = \ker A.
\]
From Theorem 1 one obtains:

**Proposition 2.** Let $x \in X_1$. If $E(x|_S) \neq \emptyset$ then $\|A\| = 1$,
\[
d(x, S^\perp) = \|Ax\| = \|x|_S\|
\]
and
\[
P_{S^\perp}(x) = x - E(Ax) = x - E(x|_S).
\]
Consequently, if $X_2$ has the extension property with respect to $X_1$ then
$\|A\| = 1$, the space $S^\perp$ is proximinal in $X_1$ and the formulae (14), (15) hold.

**Proof.** Suppose $E(x|_S) \neq \emptyset$. Taking $y \in E(x|_S)$ we have $z_0 := x - y \in S^\perp$, and
\[
\|x - z_0\| = \|y\| = \|x|_S\| = \|Ax\| = \|A(x - z_0)\|
\]
showing that $\|A\| = 1$ and that condition (7) holds. By Theorem 1, $z_0 \in P_{S^\perp}(x)$ and (14) and (15) follow from (4) and (6), respectively. □
2.1. Hahn-Banach extensions

Let $X$ be a normed space and $Y$ a closed subspace of $X$. Put $X_1 = X^*$ and $X_2 = Y^*$. By Hahn-Banach theorem every $y^* \in Y^*$ has a norm preserving extension in $X^*$, i.e. the space $Y^*$ has the extension property with respect to $X^*$. By Proposition 1, it follows that $Y^\perp$ is proximinal in $X^*$, 

(16) \quad d(x^*, Y^\perp) = \|x^*|_Y\| \quad \text{and} \quad P_{Y^\perp}(x^*) = x^* - E(x^*|_Y).

From the second formula in (16) follows Phelps’ result \[38\] that $Y^\perp$ is Chebyshevian in $X^*$ if and only if every $y^* \in Y^*$ has a unique norm-preserving extension in $X^*$, as well as the result of Xu Ji Hong \[20\], asserting that $P_{Y^\perp}(x^*)$ has affine dimension at most $k - 1$, for every $x^* \in X^*$, if and only if every $y^* \in Y^*$ has at most $k$ linearly independent norm-preserving extensions in $X^*$. For other results concerning the unicity in Hahn-Banach extension theorem see E. Oja’s papers \[34, 36\] and the monograph \[35\].

If $E$ is a Banach space with the binary intersection property then, by a result of L. Nachbin \[33\], the space $L(Y, E)$ has the extension property with respect to $L(X, E)$. Here $X, Y$ are normed spaces with $Y \subset X$ and $L(X, E) (L(Y, E))$ denotes the space of all continuous linear operators from $X$ (respectively $Y$) to $E$. It follows that the space $Y^\perp = \{ A \in L(X, E) : A|_Y = 0 \}$ is proximinal in $L(X, E)$ and the formulae (14) and (15) apply.

Using some extension results for bounded bilinear functionals and operators on 2-normed spaces one can prove similar duality results for spaces of bounded bilinear operators or functionals on 2-normed spaces (see \[6, 7\]). Let $(X, \| \cdot \|, \| \cdot \|)$ be a 2-normed space in the sense of S. Gähler \[16\] and let $E$ be a normed space. A bilinear operator $A : X_1 \times X_2 \to E$, $X_1, X_2$ subspaces of $X$, is called bounded (or Lipschitz) if $\| A(x_1, x_2) \| \leq \| x_1, x_2 \|_{X_1 \times X_2}$, for some $L \geq 0$. Denote by $L_2(X_1 \times X_2, E)$ the space of bounded bilinear operators from $X_1 \times X_2$ to $E$, and let $L_2(X_1 \times X_2) = L_2(X_1 \times X_2, \mathbb{K})$ be the space of bounded bilinear functionals on $X_1 \times X_2$. If $Z$ is a subspace of $X$ and $|b| = \mathbb{K}b$ is the subspace generated by an element $b \in X$, $b \not= 0$, then every bilinear functional $f \in L_2(Z \times |b|)$ admits a norm-preserving extension $F \in L_2(X \times |b|)$. If $E$ has the binary intersection property, then a similar extension result is valid for the spaces $L_2(Z \times |b|, E)$ and $L_2(X \times |b|, E)$ (see \[2\] or \[7\]). Denoting by $E(f)$ the set of all these extensions and by $Z^\perp = \{ F \in L_2(X \times |b|) : F|_{Z \times |b|} = 0 \}$ the annihilator of $Z \times |b|$ in $L_2(X \times |b|)$, it follows that $Z^\perp$ is proximinal in $X \times |b|$ and that 

$$
\begin{align*}
d(F, Z^\perp) &= \|F|_{Z \times |b|}\| \quad \text{and} \quad P_{Z^\perp}(F) = F - E(F|_{Z \times |b|})
\end{align*}
$$

(see \[7\]). If $E$ has the binary intersection property then the above results hold for the spaces of bounded bilinear operators $L_2(Z \times |b|, E)$ and $L_2(X \times |b|, E)$ (see \[6\]).
2.2. Helly extensions

Let $X$ be a real normed space and $J : X \to X^{**}$ the canonical embedding operator of $X$ in its bidual, defined by

$$J(x)(x^*) = x^*(x), \ x^* \in X^*.$$ 

Put $\hat{x} = J(x)$.

Let $Y$ be a closed subspace of $X$ and $Y^\perp$ its annihilator in $X^*$. For $x^{**} \in X^{**}$ define the set of Helly extensions of $x^{**}$ by

$$E(x^{**}|Y) = \left\{ y \in X : \hat{y}|Y = x^{**}|Y \quad \text{and} \quad \|\hat{y}\| = \|x^{**}|Y\| \right\}.$$ 

Helly extensions can not exist, i.e. it is possible that $E(x^{**}|Y) = \emptyset$. If $x_1^{**}, \ldots, x_n^{**}$ are in $X^{**}$ and $\epsilon > 0$ then there is $x \in X$ such that $\|x\| < \|x^{**}\| + \epsilon$ and $x_i^{**}(x) = x^{**}(x_i^*)$, $i = 1, \ldots, n$. This is Helly’s theorem (see [13, p. 86]) justifying the denomination “Helly extension”. Restricting to $J(X)$ we have

$$E(\hat{x}|Y) = \left\{ y \in X : \hat{y}|Y = \hat{x}|Y \quad \text{and} \quad \|\hat{y}\| = \|\hat{x}|Y\| \right\}$$

for $x \in X$.

Observe that if $x \in X$ is fixed and $y \in Y$ is arbitrary then, denoting by $B^*$ the closed unit ball of $X^*$, we have

$$\|x - y\| = \|\hat{x} - \hat{y}\| = \sup\{|x^*(x - y)| : x^* \in B^*\}$$

$$\geq \sup\{|y^*(x - y)| : y^* \in B^* \cap Y^\perp\}$$

$$= \sup\{|y^*(x)| : y^* \in B^* \cap Y^\perp\} = \|\hat{x}|Y\|,$$

showing that

$$d(x, Y) \geq \|\hat{x}|Y\|.$$ 

By a theorem of Hahn (see [13, Lemma II.3.12]), there exists $y_0^* \in Y^\perp$ such that $\|y_0^*\| = 1$ and $y_0^*(x) = d(x, Y)$, implying

$$d(x, Y) = y_0^*(x) = \hat{x}(y_0^*) \leq \|\hat{x}\|.$$ 

Consequently

(17) \quad $$d(x, Y) = \|\hat{x}|Y\|$$

for every $x \in X$.

Let $W := \{\hat{x}|Y : x \in X\}$ and let $A : J(X) \to W$ be the restriction operator, defined by $A\hat{x} = \hat{x}|Y$, \quad $x \in X$.

Since $Y$ is a closed subspace of $X$, it follows that for every $x \in X \setminus Y$ there exists $y^* \in Y^\perp$ such that $y^*(x) = 1$ (see [13, Consequence II.3.13]), implying

$$\ker A = \{x \in X : \hat{x}|Y = 0\} = \{x \in X : y^*(x) = 0, \ \forall y^* \in Y^\perp\} = Y.$$ 

Also, by (17) and Proposition 1,

$$d(x, \ker A) = d(x, Y) = \|\hat{x}|Y\| = \|Ax\|.$$ 

It follows that

$$P_Y(x) = x - E(\hat{x}|Y)$$
and that \( Y \) is proximinal if and only if every element \( x \in X \) admits a Helly extension. For results of this kind see \([37, 11]\).

2.3. The spaces \( c_0 \) and \( l_\infty \)

We illustrate the above considerations on the case of spaces \( c_0 \) and \( l_\infty \). As usual, denote by \( c_0 \) (\( l_\infty \)) the space of all converging to zero (respectively bounded) sequences of real numbers. Equipped with the sup-norms they are Banach spaces and \( c_0 \subseteq l_\infty \).

**Proposition 3.**

1. The subspace \( c_0 \) is proximinal in \( l_\infty \) and the distance of an element \( x \in l_\infty \) to \( c_0 \) is given by the formula

\[
d(x, c_0) = \limsup |x(n)|.
\]

2. Every continuous linear functional \( y^* \in c_0^* \) has a unique norm-preserving extension \( x^* \in l_\infty^* \).

3. The annihilator \( c_0^\perp \) of \( c_0 \) is a Chebyshev subspace of \( l_\infty^* \) and

\[
d(x^*, c_0^\perp) = \|x^*|_{c_0}\|
\]

for every \( x^* \in l_\infty^* \).

**Proof.** 1. The proof is immediate (see e.g. [4] for this result as well as for other distance formulae and proximinality results in Banach spaces of vector-valued sequences).

2. Let \( y^* \in c_0^* \), \( y^* \neq 0 \). Since \( c_0^* = l_1 \) there exists \( (a_n) \in l_1 \) such that

\[
y^*(y) = \sum_{i=1}^{\infty} a_i y(i), \forall y \in c_0, \quad \text{and} \quad \|y^*\| = \sum_{i=1}^{\infty} |a_i|.
\]

Let \( x^* \in l_\infty^* \) be such that

\[
x^*|_{c_0} = y^* \quad \text{and} \quad \|x^*\| = \|y^*\|.
\]

To prove the unicity of \( x^* \) we shall follow the ideas in the proof of Helly’s one step extension theorem (see the proof of Theorem II.3.20 in [13]).

Let \( x \in l_\infty \setminus c_0 \). For \( z \in c_0 \) we have

\[
x^*(x) - y^*(z) = x^*(x - z) \leq \|x^*\| \|x - z\| = \|y^*\| \|x - z\|
\]

implying

\[
x^*(x) \leq y^*(z) + \|x - z\|, \quad \forall z \in c_0.
\]

Similarly

\[
y^*(y) - x^*(x) = x^*(x - y) \leq \|x^*\| \|x - y\| = \|y^*\| \|x - y\|
\]

implies

\[
y^*(y) - \|y^*\| \|x - y\| \leq x^*(x), \quad \forall y \in c_0.
\]
The inequalities (22) and (23) yield
\[(24) \quad \sup_{y \in c_0} \left[ y^*(y) - \|y^*\| \|x - y\| \right] \leq x^*(x) \leq \inf_{z \in c_0} \left[ y^*(z) + \|y^*\| \|x - z\| \right].\]

Now, by (20), the inequalities (22) and (23) give
\[\sum_{i=1}^{\infty} a_i y(i) - \|x - y\| \leq \sum_{i=1}^{\infty} a_i z(i) + \|x - z\| \sum_{i=1}^{\infty} |a_i|.
\]

Writing \(|a_i| = a_i \epsilon_i\), one obtains
\[\sum_{i=1}^{\infty} |a_i| (\epsilon_i y(i) - \|x - y\|) \leq \sum_{i=1}^{\infty} |a_i| (\epsilon_i z(i) + \|x - z\|)
\]
or equivalently
\[(25) \quad \sum_{i=1}^{\infty} |a_i| \left( \epsilon_i [y(i) - x(i)] - \|x - y\| \right) \leq \sum_{i=1}^{\infty} |a_i| \left( \epsilon_i [z(i) - x(i)] + \|x - z\| \right)
\]
for all \(y, z \in c_0\). Since \(\epsilon_i [y(i) - x(i)] - \|x - y\| \leq 0\), for all \(i \in \mathbb{N}\), it follows that the supremum for \(y \in c_0\) in the left-hand side of (25) is \(\leq 0\).

Let \(\beta = \|x\| > 0\), and let \(y_n(i) = x(i) + \epsilon_i (\beta + 1)\), for \(1 \leq i \leq n\), and \(y_n(i) = 0\), for \(i > n\), \(n \in \mathbb{N}\). Then \(\|x - y_n\| = \beta + 1\) for \(n\) sufficiently large (such that at least one \(a_i, 1 \leq i \leq n\), be different from zero), so that the expression in the left-hand side of (25) becomes
\[\left| \sum_{i=1}^{\infty} |a_i| (\epsilon_i [y_n(i) - x(i)] - \|x - y_n\|) \right| \leq \sum_{i>n} |\epsilon_i x(i) + \beta + 1| \leq (2\beta + 1) \sum_{i>n} |a_i| \to 0, \quad n \to \infty.
\]

It follows that
\[\sup_{y \in c_0} \left\{ \sum_{i=1}^{\infty} (\epsilon_i [y(i) - x(i)] - \|x - y\|) \right\} = 0
\]
or equivalently
\[\sup_{y \in c_0} \left\{ \sum_{i=1}^{\infty} |a_i| (\epsilon_i y(i) - \|x - y\|) \right\} = \sum_{i=1}^{\infty} |a_i| \epsilon_i x(i) = \sum_{i=1}^{\infty} a_i x(i).
\]

Reasoning similarly, one obtains
\[\inf_{z \in c_0} \left\{ \sum_{i=1}^{\infty} |a_i| (\epsilon_i z(i) + \|x - z\|) \right\} = \sum_{i=1}^{\infty} a_i x(i).
\]

Taking into account the inequalities (24) and the fact that \(x \in l_\infty\) was arbitrarily chosen, it follows
\[x^*(x) = \sum_{i=1}^{\infty} a_i x(i)\]
for all \( x \in l_\infty \), proving the unicity of the extension \( x^* \).

\[ \square \]

**Remark.** The above proof of the unicity of the extension of linear functionals on \( c_0 \) is suggested in [22] Problem 12.20.

\[ \square \]

Using the representation of continuous linear functionals on \( l_\infty \) we can obtain more information on the behavior of \( c_0^\perp \) in \( l_\infty^* \). It is known that the dual space of \( l_\infty \) can be identified with the space \( ba(\mathcal{P}(N)) \) of all finitely additive bounded measures on \( \mathcal{P}(N) \) (see [13, Th.IV.8.16] or [2, C. 4.7.11]). Following [2] we shall call the elements of \( ba(\mathcal{P}(N)) \) charges. The variation of a charge \( \mu \in ba(\mathcal{P}(N)) \) is defined by

\[
|\mu|(A) = \sup \left\{ \sum_{i=1}^{n} |\mu(A_i)| : n \in N, A = \bigcup_{i=1}^{n} A_i, \text{ with } A_i \text{ pairwise disjoint} \right\}.
\]

One shows that \( |\mu| \) is in \( ba(\mathcal{P}(N)) \) too, and

\[
\|\mu\| = |\mu|(N)
\]

is a norm on \( ba(\mathcal{P}(N)) \) with respect to which \( ba(\mathcal{P}(N)) \) is a Banach space. The space \( ca(\mathcal{P}(N)) \) of countably additive finite measures on \( \mathcal{P}(N) \) is a closed subspace of \( ba((N)) \) and the correspondence

\[
\lambda \mapsto \left( \lambda(\{i\}) ; i \in \mathbb{N} \right), \quad \lambda \in ca(\mathcal{P}(N)),
\]

is an isometric isomorphism between the Banach spaces \( ca(\mathcal{P}(N)) \) and \( l_1 \).

For \( \mu \in ba(\mathcal{P}(N)) \) the formula

\[
x^*_\mu(x) = \int x \cdot d\mu, \quad x \in l_\infty
\]

defines a continuous linear functional \( x^*_\mu \) on \( l_\infty \) with \( \|x^*_\mu\| = \|\mu\| \). We shall denote this functional simply by \( \mu \). If \( x \in c_0 \) then the sequence \( x_n = (x(1), \ldots, x(n), 0, \ldots) \), \( n \in \mathbb{N} \), converges uniformly to \( x \) so that, by the definition of the integral with respect to \( \mu \),

\[
\mu(x) = \lim_{n \to \infty} \mu(x_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(\{i\})x(i)
\]

i.e.

\[
\mu(x) = \sum_{i=1}^{\infty} \mu(\{i\})x(i), \quad \forall x \in c_0.
\]

(26)

A charge \( \mu \in ba(\mathcal{P}(N)) \) is called purely finitely additive (or a pure charge) if \( 0 \leq \lambda \leq |\mu| \) implies \( \lambda = 0 \), for every \( \lambda \in ca(\mathcal{P}(N)) \). Denote by \( pba(\mathcal{P}(N)) \) the set of all purely finitely additive measures. By Yosida-Hewitt theorem (see [13] or [2, p. 240]), every charge \( \mu \in ba(\mathcal{P}(N)) \) admits a unique decomposition, called Yosida-Hewitt decomposition, of the form

\[
\mu = \mu_c + \mu_p
\]

(27)

with \( \mu_c \) countably additive and \( \mu_p \) purely finitely additive.
In our case
\begin{equation}
\mu_c(A) = \sum_{i \in A} \mu(\{i\})
\end{equation}
and
\begin{equation}
\mu_p(A) = \mu(A) - \sum_{i \in A} \mu(\{i\})
\end{equation}
for every $A \subset \mathbb{N}$. It follows that $\mu_p(A) = 0$ for every finite subset $A$ of $\mathbb{N}$. In particular $\mu(\{i\}) = 0$, $i \in \mathbb{N}$, so that, by (28), $\mu_p(x) = 0$, $x \in c_0$, showing that
\[ pba(\mathcal{P}(\mathbb{N})) \subset c_0. \]
Conversely, if $\mu \in c_0$ then, by (26),
\[ \sum_{i=1}^{\infty} \mu(\{i\})x(i) = \mu(x) = 0, \quad \forall x \in c_0, \]
implying $\mu(\{i\}) = 0$, $i \in \mathbb{N}$, which by (28) yields $\mu_c = 0$ and $\mu = \mu_p \in pba(\mathcal{P}(\mathbb{N}))$, i.e. $c_0^\perp \subset pba(\mathcal{P}(\mathbb{N}))$.
Consequently
\begin{equation}
\frac{c_0}{\mu_c} = pba(\mathcal{P}(\mathbb{N})).
\end{equation}
If $\mu = \mu_c + \mu_p$ then, by (28),
\[ \|\mu_c\| = \sum_{i=1}^{\infty} |\mu(\{i\})|. \]
By (29)
\[ |\mu_p(A_1)| + \ldots + |\mu_p(A_k)| = |\mu(A_1)| + \ldots + |\mu_p(A_k)| - \sum_{i=1}^{\infty} |\mu(\{i\})| \]
for every decomposition $\mathbb{N} = A_1 \cup \ldots \cup A_k$ of $\mathbb{N}$ into pairwise disjoint sets. It follows $\|\mu_p\| = |\mu| - |\mu_c|$ or, equivalently,
\begin{equation}
\|\mu\| = |\mu_c| + |\mu_p|.
\end{equation}
Consequently
\begin{equation}
\text{ba}(\mathcal{P}(\mathbb{N})) = \text{ca}(\mathcal{P}(\mathbb{N})) \oplus \text{pba}(\mathcal{P}(\mathbb{N})) = \text{ca}(\mathcal{P}(\mathbb{N})) \oplus c_0^\perp.
\end{equation}
A closed subspace $Y$ of a Banach space $X$ is called an M-ideal if $X^*$ can be decomposed into a direct sum
\[ X^* = Y^\perp \oplus \hat{Y} \]
with $\|x^*\| = \|y^*\| + \|z^*\|$, for every $(x^*, y^*, z^*) \in X^* \times Y^\perp \times \hat{Y}$, such that $x^* = y^* + z^*$. M-ideals were considered first by E. Alfsen and Efros [1]. For a thorough exposition of the present day situation in M-ideals theory we recommend the monograph [17] (see also [35]). In this language, the relations (31) and (32) tell us that $c_0$ is an M-ideal in $l_\infty$. From the general theory
of M-ideals it follows that the space $c_0$ is proximinal in $l_\infty$ and that every continuous linear functional on $c_0$ has a unique norm-preserving extension to $l_\infty$. Since $\mu_p|c_0 = 0$, for $\mu \in ba(P(\mathbb{N}))$, the formulae (16) become

$$d(\mu, c_0^\perp) = \|\mu|c_0\| = \|\mu_c\| = \sum_{i=1}^{\infty} |\mu(\{i\})|$$

and

$$P_{c_0^\perp}(\mu) = \{\mu_p\}.$$

This is another way to obtain the results in Proposition 2.

Since $c_0$ is proximinal in $l_\infty$ it follows that every $x \in l_\infty$ admits a Helly extension with respect to $c_0$. To obtain concrete representations in this situation we have to work with the bidual $l_\infty^{**} = ba^*(P(\mathbb{N}))$. But, as it is asserted in [2, p. 231], no satisfactory representation for the elements of $ba^*(P(\mathbb{N}))$ is known.

2.4. Tietze extensions

Let $T$ be a locally compact Hausdorff topological space and $S$ a nonvoid closed subset of $T$. Denote by $C_0(T)$ ($C_0(S)$) the space of all real-valued continuous functions on $T$ (respectively $S$) vanishing at infinity. Equipped with the sup-norms $C_0(T)$ and $C_0(S)$ are Banach algebras and

$$Z(S) = \{ F \in C_0(T) : F|S = 0 \}$$

is a closed ideal in $C_0(T)$. By Tietze extension theorem (see [14]) every $f \in C_0(S)$ admits a norm-preserving extension $F \in C_0(T)$. It follows that Proposition 1 can be applied to deduce that $Z(S)$ is a proximinal subspace of $C_0(T)$ and that the formulae (33)

$$d(F, Z(S)) = \|F|S\| \quad \text{and} \quad P_{Z(S)}(F) = F - E(F|S)$$

hold.

Results of this kind have been obtained in [10, 37].

Using Dugundji’s [12] vector-version of Tietze extension theorem one obtains the validity of the formulae (33) for $C_0(T, E)$, $C_0(S, E)$, with $E$ a Banach space.

2.5. Spaces of Lipschitz and Hölder functions

Let $(X, d)$ be a metric space. A function $F : X \to \mathbb{R}$ is called Lipschitz if

$$(34) \quad \|F\| := \sup \left\{ \frac{|F(x_1) - F(x_2)|}{d(x_1, x_2)} : x_1, x_2 \in X, \ x_1 \neq x_2 \right\} < \infty.$$ 

The quantity $\|F\|$ is called the Lipschitz norm of the function $F$ and the space of all real-valued Lipschitz functions on $X$ is denoted by Lip$(X)$. Since $\|F\| = 0$ for constant functions, (34) is in fact only a semi-norm on Lip$(X)$. There are several ways to transform Lip$(X)$ into a Banach space. One consists in fixing a point $x_0 \in X$ and consider the space Lip$0(X)$ of all functions in Lip$(X)$ vanishing at $x_0$. Other way consists in considering the space BLip$(X)$.
of all bounded functions in \( \text{Lip}(X) \), normed by
\[
\|F\|_1 = \|F\| + \|F\|_\infty
\]
or by
\[
\|F\|_2 = \max\{\|F\|, \|F\|_\infty\},
\]
where \( \|F\| \) is given by (34) and \( \|F\|_\infty \) is the sup-norm.

McShane [25] proved that every Lipschitz functions \( f \), defined on a subset \( Y \) of \( X \), admits a norm preserving extension to \( X \) (see also [8, 26]). The case \( X = \mathbb{R}^n \) was considered by M. Kirszbraun [22]. Based on this result one can show that the space \( \text{Lip}_0(Y) \) has the extension property in \( \text{Lip}_0(X) \) so that, by Proposition 1,
\[
d(F, Y^\perp) = \|f|_Y\| \quad \text{and} \quad P_{Y^\perp}(F) = F - E(F|_Y)
\]
for every \( F \in \text{Lip}_0(X) \), where
\[
Y^\perp = \{ F \in \text{Lip}_0(X) : F|_Y = 0 \}.
\]
Here \( Y \) is a subset of \( X \) containing the fixed point \( x_0 \).

Similar results hold for the spaces \( \text{BLip}(Y) \) and \( \text{BLip}(X) \) with respect to the norms (35) or (36) (see [29]). In this case one can show that every \( f \in \text{BLip}(Y) \) has an extension \( F \in \text{BLip}(X) \) preserving both the Lipschitz and uniform norms, implying that the space \( \text{BLip}(Y) \) have the extension property with respect to \( \text{BLip}(X) \) for both of the norms (35) and (36).

For \( 0 < \alpha \leq 1 \), a function \( F : X \to \mathbb{R} \) is called \textit{Hölder of order} \( \alpha \) if
\[
\|F\|_\alpha := \sup \left\{ \frac{|F(x_1) - F(x_2)|}{d^\alpha(x_1, x_2)} : x_1, x_2 \in X, \ x_1 \neq x_2 \right\} < \infty.
\]
Denote by \( \Lambda^\alpha(X) \) the space of all Hölder functions on \( X \). To obtain Banach spaces of Hölder functions one can proceed like above, by considering the space \( \Lambda^\alpha_0(X) \) of Hölder functions vanishing at a fixed point \( x_0 \) or the space \( \Lambda^\alpha(X) \) of bounded Hölder functions on \( X \). Duality results for these spaces have been obtained by C. Mustaţa [31, 32].

REFERENCES


13 Phelps type duality results in best approximation


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