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PHELPS TYPE DUALITY RESULTS IN BEST APPROXIMATION

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Abstract. The aim of the present paper is to show that many Phelps type duality result, relating the extension properties of various classes of functions (continuous, linear continuous, bounded bilinear, Hölder-Lipschitz) with the approximation properties of some annihilating spaces, can be derived in a unitary and simple way from a formula for the distance to the kernel of a linear operator, extending the well-known distance formula to hyperplanes in normed spaces. The case of spaces c_0 and l^∞ is treated in details.

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THE DISTANCE FORMULA

Let X be a normed space (over $\mathbb{R}$ or $\mathbb{C}$) and Y a closed subset of X. For $x \in X$ put

$$d(x,Y) = \inf\{\|x-y\| : y \in Y\},$$
$$P_Y(x) = \{y \in Y : \|x-y\| = d(x,Y)\}.$$

The quantity $d(x,Y)$ is the *distance* from x to Y and the elements in $P_Y(x)$ are called *nearest points* (or *elements of best approximation*) for x in Y. The set-valued map P_Y is called the *metric projection.* The subspace Y is called *proximinal* if $P_Y(x) \neq \emptyset$, for every $x \in X$, *Chebyshevian* if $P_Y(x)$ is a singleton for every $x \in X$, and *antiproximinal* if $P_Y(x) = \emptyset$, for every $x \in X\setminus Y$ (observe that $P_Y(y) = \{y\}$, for $y \in Y$).

Denote by X^* the conjugate space of X and let

$$Y^\perp = \{x^* \in X^* : x^*|_Y = 0\} \tag{1}$$

be the *annihilator* of Y in X^*. In the seminal paper [38] R. R. Phelps initiated the study of the relations between the extension properties of the space Y and the approximation properties of its annihilator $Y^\perp$. Namely, $Y^\perp$ is Chebyshevian if and only if every functional $y^* \in Y^*$ has a unique norm-preserving extension $x^* \in X^*$. It is known that, by Hahn-Banach theorem, every $y^* \in Y^*$ has at least one norm-preserving extension. Since then, there have been found a lot of situations in which similar duality results hold, corresponding to variuos extension results – Helly extension theorem for linear functionals, Tietze's extension theorem for continuous functions, McShane's extension theorem for

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Lipschitz functions, Nachbin's extension theorem for continuous linear operators etc.

The aim of the present paper is to show that all these results follow immediately from a formula for the distance to the kernel of a continuous linear operator, inspired by the well-known distance formula to hyperplanes in normed spaces.

For a continuous linear operator $A : X_1 \to X_2,$ between two normed spaces X_1, X_2, let

$$Z = \{x \in X_1 : Ax = 0\} \tag{2}$$

be its kernel. Also, for $x \in X_1$, put

$$E(x) = \Big\{y \in X_1 : Ay = Ax \quad \text{and} \quad \|y\| = \tfrac{\|Ax\|}{\|A\|}\Big\}. \tag{3}$$

THEOREM 1. 1. $d(x, Z) \geq \frac{\|Ax\|}{\|A\|}.$

2. *We have*

$$d(x, Z) = \tfrac{\|Ax\|}{\|A\|} \tag{4}$$

if and only if there exists a sequence (z_n) *in* Z *such that*

$$\|x - z_n\| \to \tfrac{\|Ax\|}{\|A\|}. \tag{5}$$

3 (a). *If* (4) *holds then*

$$P_Z(x) = x - E(x) \tag{6}$$

(*with the convention* $x - \emptyset = \emptyset$).

3 (b). *If there is* $z_0 \in Z$ *such that*

$$\|x - z_0\| = \tfrac{\|Ax\|}{\|A\|} \tag{7}$$

then $z_0 \in P_Z(x)$ *and the formulae* (4) *and* (6) *hold.*

Proof. 1. For every $z \in Z$, we have

$$\|Ax\| = \|A(x - z)\| \leq \|A\|\|x - z\|$$

showing that (3) holds.

2. Let $(z_n) \subset Z$ verifying (5). Then

$$d(x, Z) \leq \|x - z_n\|, \; \forall n \in \mathbb{N},$$

and, letting $n \to \infty,$ one obtains

$$d(x, Z) \leq \tfrac{\|Ax\|}{\|A\|}$$

which, combined with the point 1 of the theorem, yields (4).

Conversely, if the equality (4) holds and (z_n) is a sequence in Z such that $\|x - z_n\| \to d(x, Z),$ then the sequence (z_n) verifies (5).

3. (a) Follows from the equivalences:

$$z \in P_Z(x) \Longleftrightarrow z \in Z \quad \text{and} \quad \|x - z\| = d(x, Z) = \tfrac{\|Ax\|}{\|A\|}$$
$$\Longleftrightarrow x - z \in E(x) \Longleftrightarrow z \in x - E(x).$$

3. (b) Observe that the equality (7) implies that (5) holds with $z_n = z_0$, $n = 1, 2, \ldots$, so that, by the point 2 of the theorem, (4) and (6) hold too. Since

$$\|x - z_0\| = \tfrac{\|Ax\|}{\|A\|} = d(x, Z)$$

it follows that $z_0 \in P_Z(x)$. □

Remark. If $y \in X_1$ is fixed and $W := y + Z$ then

$$d(x, W) = d(x - y, Z) = \tfrac{\|Ax - Ay\|}{\|A\|}. \tag{8}$$ □

Examples

1. *The distance from a point to a hyperplane*

Let X be a normed space, $x^* \in X^*$, $x^* \neq 0$, and $Z = \ker x^*$. Take

$$A := x^* : X \to \mathbb{K}$$

($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

Let $x \in X$. First show that condition (5) is always fulfilled. Indeed, if $u_n \in X$, $\|u_n\| = 1$, $n \in \mathbb{N}$, are such that $|x^*(u_n)| \to \|x^*\|$, then

$$z_n := x - \frac{x^*(x)}{x^*(u_n)} u_n \in Z \quad \text{and} \quad \|x - z_n\| = \frac{|x^*(x)|}{|x^*(u_n)|} \to \frac{|x^*(x)|}{\|x^*\|}.$$

Therefore

$$d(x, Z) = \frac{|x^*(x)|}{\|x^*\|}$$

for every $x \in X$.

Observe now that condition (7) holds if and only if x^* supports the closed unit ball B of X. Indeed, if $u_0 \in X$, $\|u_0\| = 1$, is such that $|x^*(u_0)| = \|x^*\|$, then $z_0 := x - (x^*(x)/x^*(u_0))\, u_0$ is in Z and

$$\|x - z_0\| = \frac{|x^*(x)|}{|x^*(u_0)|} = \frac{|x^*(x)|}{\|x^*\|}.$$

Conversely, if, for some $x_0 \in X$, there is an element $z_0 \in Z$ such that $\|x - z_0\| = |x^*(x_0)|/\|x^*\|$, then

$$|x^*(x_0 - z_0)| = |x^*(x)| = \|x^*\|\|x_0 - z_0\|,$$

showing that x^* attains its norm on the element $u_0 = (x_0 - z_0)/\|x_0 - z_0\|$ of B.

In fact we have shown that if x^* supports the unit ball B of X then (7) holds for every $x \in X$, and if (7) holds for a single element $x_0 \in X \setminus Z$ then

x^* supports the unit ball B and, therefore, (7) holds for every $x \in X$. It follows that the subspace $Z = \ker x^*$ is proximinal if x^* supports the unit ball of X, and antiproximinal if not.

If $h_0 \in X$ and $H = h_0 + Z = \{x \in X : x^*(x) = a\}$ where $a = x^*(h_0)$, is a closed hyperplane paralel to Z then, by (8),

$$d(x, H) = \frac{|x^*(x) - a|}{\|x^*\|}$$

a well-known formula.

2. *Restriction operators*

Let E be a normed space and S, T nonvoid sets with $S \subset T$. Consider two normed spaces $X_1 = X_1(T, E)$ and $X_2 = X_2(S, E)$ of mappings from T (respectively S) to E, the vector operations being defined pointwise. Suppose that there are verified the following conditions

$$x|_S \in X_2 \quad \text{and} \quad \|x|_S\| \leq \|x\| \tag{9}$$

for every $x \in X_1$. For $y \in X_2$ denote by

$$E(y) = \{x \in X_1 : x|_S = y \quad \text{and} \quad \|x\| = \|y\|\} \tag{10}$$

the (possibly empty) set of norm-preserving extensions of y in X_1. One says that the space X_2 has *the extension property* with respect to X_1 if $E(y) \neq \emptyset$, for every $y \in X_2$. Let $A : X_1 \to X_2$ be the *restriction operator* defined by

$$Ax = x|_S, \quad x \in X_1. \tag{11}$$

By (9), A is well defined, linear, continuous, and

$$\|A\| \leq 1. \tag{12}$$

Put

$$S^\perp = \{x \in X_1 : x|_S = 0\} = \ker A. \tag{13}$$

From Theorem 1 one obtains:

Proposition 2. *Let $x \in X_1$. If $E(x|_S) \neq \emptyset$ then $\|A\| = 1$,*

$$d(x, S^\perp) = \|Ax\| = \|x|_S\| \tag{14}$$

and

$$P_{S^\perp}(x) = x - E(Ax) = x - E(x|_S). \tag{15}$$

Consequently, if X_2 has the extension property with respect to X_1 then $\|A\| = 1$, the space $S^\perp$ is proximinal in X_1 and the formulae (14), (15) *hold.*

Proof. Suppose $E(x|_S) \neq \emptyset$. Taking $y \in E(x|_S)$ we have $z_0 := x - y \in S^\perp$, and

$$\|x - z_0\| = \|y\| = \|x|_S\| = \|Ax\| = \|A(x - z_0)\|$$

showing that $\|A\| = 1$ and that condition (7) holds. By Theorem 1, $z_0 \in P_{S^\perp}(x)$ and (14) and (15) follow from (4) and (6), respectively. □

2.1. *Hahn-Banach extensions*

Let X be a normed space and Y a closed subspace of X. Put $X_1 = X^*$ and $X_2 = Y^*$. By Hahn-Banach theorem every $y^* \in Y^*$ has a norm preserving extension in X^*, i.e. the space Y^* has the extension property with respect to X^*. By Proposition 1, it follows that $Y^\perp$ is proximinal in X^*,

$$d(x^*, Y^\perp) = \|x^*|_Y\| \quad \text{and} \quad P_{Y^\perp}(x^*) = x^* - E(x^*|_Y). \tag{16}$$

From the second formula in (16) follows Phelps' result [38] that $Y^\perp$ is Chebyshevian in X^* if and only if every $y^* \in Y^*$ has a unique norm-preserving extension in X^*, as well as the result of Xu Ji Hong [20], asserting that $P_{Y^\perp}(x^*)$ has affine dimension at most $k-1$, for every $x^* \in X^*$, if and only if every $y^* \in Y^*$ has at most k linearly independent norm-preserving extensions in X^*. For other results concerning the unicity in Hahn-Banach extension theorem see E. Oja's papers [34, 36] and the monograph [35].

If E is a Banach space with the binary intersection property then, by a result of L. Nachbin [33], the space $L(Y, E)$ has the extension property with respect to $L(X, E)$. Here X, Y are normed spaces with $Y \subset X$ and $L(X, E)$ $(L(Y, E))$ denotes the space of all continuous linear operators from X (respectively Y) to E. It follows that the space $Y^\perp = \{A \in L(X, E) : A|_Y = 0\}$ is proximinal in $L(X, E)$ and the formulae (14) and (15) apply.

Using some extension results for bounded bilinear functionals and operators on 2-normed spaces one can prove similar duality results for spaces of bounded bilinear operators or functionals on 2-normed spaces (see [6, 7]) Let $(X, \|\,,\,\|)$ be a 2-normed space in the sense of S. Gähler [16] and let E be a normed space. A bilinear operator $A : X_1 \times X_2 \to E$, X_1, X_2 subspaces of X, is called bounded (or Lipschitz) if $\|A(x_1, x_2)\| \le \|x_1, x_2\|$, $(x_1, x_2) \in X_1 \times X_2$, for some $L \ge 0$. Denote by $L_2(X_1 \times X_2, E)$ the space of bounded bilinear operators from $X_1 \times X_2$ to E, and let $L_2(X_1 \times X_2) = L_2(X_1 \times X_2, \mathbb{K})$ be the space of bounded bilinear functionals on $X_1 \times X_2$. If Z is a subspace of X and $[b] = \mathbb{K}b$ is the subspace generated by an element $b \in X$, $b \ne 0$, then every bilinear functional $f \in L_2(Z \times [b])$ admits a norm-preserving extension $F \in L_2(X \times [b])$. If E has the binary intersection property, then a similar extension result is valid for the spaces $L_2(Z \times [b], E)$ and $L_2(X \times [b], E)$ (see [3] or [6]). Denoting by $E(f)$ the set of all these extensions and by

$$Z_b^\perp = \{F \in L_2(X \times [b]) : F|_{Z \times [b]} = 0\}$$

the annihilator of $Z \times [b]$ in $L_2(X \times [b])$, it follows that $Z_b^\perp$ is proximinal in $X \times [b]$ and that

$$d(F, Z_b^\perp) = \|F|_{Z \times [b]}\| \quad \text{and} \quad P_{Z_b^\perp}(F) = F - E(F|_{Z \times [b]})$$

(see [7]). If E has the binary intersection property then the above results hold for the spaces of bounded bilinear operators $L_2(Z \times [b], E)$ and $L_2(X \times [b], E)$ (see [6]).

2.2. *Helly extensions*

Let X be a real normed space and $J : X \to X^{**}$ the canonical embedding operator of X in its bidual, defined by

$$J(x)(x^*) = x^*(x),\ x^* \in X^*.$$

Put $\hat{x} = J(x)$.

Let Y be a closed subspace of X and $Y^\perp$ its annihilator in X^*. For $x^{**} \in X^{**}$ define the set of *Helly extensions* of x^{**} by

$$E(x^{**}|_{Y^\perp}) = \Big\{ y \in X \ : \ \hat{y}|_{Y^\perp} = x^{**}|_{Y^\perp} \text{ and } \|\hat{y}\| = \|x^{**}|_{Y^\perp}\| \Big\}.$$

Helly extensions can not exist, i.e. it is possible that $E(x^{**}|_{Y^\perp}) = \emptyset$. If $x_1^*, \ldots, x_n^*$ are in X^* and $\epsilon > 0$ then there is $x \in X$ such that $\|x\| < \|x^{**}\| + \epsilon$ and $x_i^*(x) = x^{**}(x_i^*),\ i = 1, \ldots, n$. This is Helly's theorem (see [13, p. 86]) justifying the denomination "Helly extension". Restricting to $J(X)$ we have

$$E(\hat{x}|_{Y^\perp}) = \{ y \in X \ : \ \hat{y}|_{Y^\perp} = \hat{x}|_{Y^\perp} \text{ and } \|\hat{y}\| = \|\hat{x}|_{Y^\perp}\| \}$$

for $x \in X$.

Observe that if $x \in X$ is fixed and $y \in Y$ is arbitrary then, denoting by B^* the closed unit ball of X^*, we have

$$\begin{aligned} \|x - y\| = \|\widehat{x - y}\| &= \sup\{|x^*(x - y)| : x^* \in B^*\} \\ &\geq \sup\{|y^*(x - y)| : y^* \in B^* \cap Y^\perp\} \\ &= \sup\{|y^*(x)| : y^* \in B^* \cap Y^\perp\} = \|\hat{x}|_{Y^\perp}\|, \end{aligned}$$

showing that

$$d(x, Y) \geq \|\hat{x}|_{Y^\perp}\|.$$

By a theorem of Hahn (see [13, Lemma II.3.12]), there exists $y_0^* \in Y^\perp$ such that $\|y_0^*\| = 1$ and $y_0^*(x) = d(x, Y)$, implying

$$d(x, Y) = y_0^*(x) = \hat{x}(y_0^*) \leq \|\hat{x}\|.$$

Consequently

$$d(x, Y) = \|\hat{x}|_{Y^\perp}\| \tag{17}$$

for every $x \in X$.

Let $W := \{\hat{x}|_{Y^\perp} : x \in X\}$ and let $A : J(X) \to W$ be the restriction operator, defined by $A\hat{x} = \hat{x}|_{Y^\perp}, \quad x \in X$.

Since Y is a closed subspace of X, it follows that for every $x \in X \setminus Y$ there exists $y^* \in Y^\perp$ such that $y^*(x) = 1$ (see [13, Consequence II.3.13]), implying

$$\ker A = \{x \in X : \hat{x}|_{Y^\perp} = 0\} = \{x \in X : y^*(x) = 0,\ \forall y^* \in Y^\perp\} = Y.$$

Also, by (17) and Proposition 1,

$$d(x, \ker A) = d(x, Y) = \|\hat{x}|_{Y^\perp}\| = \|Ax\|.$$

It follows that

$$P_Y(x) = x - E(\hat{x}|_{Y^\perp})$$

and that Y is proximinal if and only if every element $x \in X$ admits a Helly extension. For results of this kind see [37, 11].

2.3. *The spaces c_0 and l_∞*

We illustrate the above considerations on the case of spaces c_0 and l_∞. As usual, denote by c_0 (l_∞) the space of all converging to zero (respectively bounded) sequences of real numbers. Equipped with the sup-norms they are Banach spaces and $c_0 \subset l_\infty$.

PROPOSITION 3. 1. *The subspace c_0 is proximinal in l_∞ and the distance of an element $x \in l_\infty$ to c_0 is given by the formula*

$$d(x, c_0) = \limsup |x(n)|. \tag{18}$$

2. *Every continuous linear functional $y^* \in c_0^*$ has a unique norm-preserving extension $x^* \in l_\infty^*$.*
3. *The annihilator $c_0^\perp$ of c_0 is a Chebyshev subspace of l_∞^* and*

$$d(x^*, c_0^\perp) = \|x^*|_{c_0}\| \tag{19}$$

for every $x^* \in l_\infty^*$.

Proof. 1. The proof is immediate (see e.g. [4] for this result as well as for other distance formulae and proximinality results in Banach spaces of vector-valued sequences).

2. Let $y^* \in c_0^*$, $y^* \neq 0$. Since $c_0^* = l_1$ there exists $(a_n) \in l_1$ such that

$$y^*(y) = \sum_{i=1}^{\infty} a_i y(i),\ \forall y \in c_0, \quad \text{and} \quad \|y^*\| = \sum_{i=1}^{\infty} |a_i|. \tag{20}$$

Let $x^* \in l_\infty^*$ be such that

$$x^*|_{c_0} = y^* \quad \text{and} \quad \|x^*\| = \|y^*\|. \tag{21}$$

To prove the unicity of x^* we shall follow the ideas in the proof of Helly's one step extension theorem (see the proof of Theorem II.3.20 in [13]).

Let $x \in l_\infty \setminus c_0$. For $z \in c_0$ we have

$$x^*(x) - y^*(z) = x^*(x - z) \le \|x^*\|\|x - z\| = \|y^*\|\|x - z\|$$

implying

$$x^*(x) \le y^*(z) + \|x - z\|, \quad \forall z \in c_0. \tag{22}$$

Similarly

$$y^*(y) - x^*(x) = x^*(x - y) \le \|x^*\|\|x - y\| = \|y^*\|\|x - y\|$$

implies

$$y^*(y) - \|y^*\|\|x - y\| \le x^*(x), \quad \forall y \in c_0. \tag{23}$$

The inequalities (22) and (23) yield

$$\sup_{y\in c_0}\Big[y^*(y)-\|y^*\|\|x-y\|\Big]\le x^*(x)\le \inf_{z\in c_0}\Big[y^*(z)+\|y^*\|\|x-z\|\Big]. \tag{24}$$

Now, by (20), the inequalities (22) and (23) give

$$\sum_{i=1}^{\infty}a_iy(i)-\|x-y\|\sum_{i=1}^{\infty}|a_i|\le\sum_{i=1}^{\infty}a_iz(i)+\|x-z\|\sum_{i=1}^{\infty}|a_i|.$$

Writing $|a_i|=a_i\epsilon_i,$ one obtains

$$\sum_{i=1}^{\infty}|a_i|\big(\epsilon_iy(i)-\|x-y\|\big)\le\sum_{i=1}^{\infty}|a_i|\big(\epsilon_iz(i)+\|x-z\|\big)$$

or equivalently

$$\sum_{i=1}^{\infty}|a_i|\Big(\epsilon_i[y(i)-x(i)]-\|x-y\|\Big)\le\sum_{i=1}^{\infty}|a_i|\Big(\epsilon_i[z(i)-x(i)]+\|x-z\|\Big) \tag{25}$$

for all $y,z\in c_0$. Since $\epsilon_i[y(i)-x(i)]-\|x-y\|\le 0,$ for all $i\in\mathbb{N}$, it follows that the supremum for $y\in c_0$ in the left-hand side of (25) is ≤ 0.

Let $\beta=\|x\|>0$, and let $y_n(i)=x(i)+\epsilon_i(\beta+1),$ for $1\le i\le n$, and $y_n(i)=0,$ for $i>n,$ $n\in\mathbb{N}$. Then $\|x-y_n\|=\beta+1$ for n sufficiently large (such that at least one $a_i,$ $1\le i\le n,$ be different from zero), so that the expression in the left-hand side of (25) becomes

$$\begin{aligned}\Big|\sum_{i=1}^{\infty}|a_i|\big(\epsilon_i[y_n(i)-x(i)]-\|x-y_n\|\big)\Big|&\le\sum_{i>n}|\epsilon_ix(i)+\beta+1|\\&\le(2\beta+1)\sum_{i>n}|a_i|\to 0,\ \text{for } n\to\infty.\end{aligned}$$

It follows that

$$\sup_{y\in c_0}\Big\{\sum_{i=1}^{\infty}\big(\epsilon_i[y(i)-x(i)]-\|x-y\|\big)\Big\}=0$$

or equivalently

$$\sup_{y\in c_0}\Big\{\sum_{i=1}^{\infty}|a_i|\big(\epsilon_iy(i)-\|x-y\|\big)\Big\}=\sum_{i=1}^{\infty}|a_i|\epsilon_ix(i)=\sum_{i=1}^{\infty}a_ix(i).$$

Reasoning similarly, one obtains

$$\inf_{z\in c_0}\Big\{\sum_{i=1}^{\infty}|a_i|\big(\epsilon_iz(i)+\|x-z\|\big)\Big\}=\sum_{i=1}^{\infty}a_ix(i).$$

Taking into account the inequalities (24) and the fact that $x\in l_\infty$ was arbitrarily chosen, it follows

$$x^*(x)=\sum_{i=1}^{\infty}a_ix(i)$$

for all $x \in l_\infty$, proving the unicity of the extension x^*. □

Remark. The above proof of the unicity of the extension of linear functionals on c_0 is suggested in [41, Problem 12.20]. □

Using the representation of continuous linear functionals on l_∞ we can obtain more information on the behavior of $c_0^\perp$ in l_∞^*. It is known that the dual space of l_∞ can be identified with the space $\mathrm{ba}(\mathcal{P}(\mathbb{N}))$ of all finitely additive bounded measures on $\mathcal{P}(\mathbb{N})$ (see [13, Th.IV.8.16] or [2, C. 4.7.11]). Following [2] we shall call the elements of $\mathrm{ba}(\mathcal{P}(\mathbb{N}))$ *charges*. The variation of a charge $\mu \in \mathrm{ba}(\mathcal{P}(\mathbb{N}))$ is defined by

$$|\mu|(A) = \sup\Big\{\sum_{i=1}^n |\mu(A_i)| : n \in \mathbb{N},\ A = \bigcup_{i=1}^n A_i,\ \text{with } A_i \text{ pairwise disjoint}\Big\}.$$

One shows that $|\mu|$ is in $\mathrm{ba}(\mathcal{P}(\mathbb{N}))$ too, and

$$\|\mu\| = |\mu|(\mathbb{N})$$

is a norm on $\mathrm{ba}(\mathcal{P}(\mathbb{N}))$ with respect to which $\mathrm{ba}(\mathcal{P}(\mathbb{N}))$ is a Banach space. The space $\mathrm{ca}(\mathcal{P}(\mathbb{N}))$ of countably additive finite measures on $\mathcal{P}(\mathbb{N}))$ is a closed subspace of $\mathrm{ba}((\mathbb{N}))$ and the correspondence

$$\lambda \mapsto \Big(\lambda(\{i\}); i \in \mathbb{N}\Big), \quad \lambda \in \mathrm{ca}(\mathcal{P}(\mathbb{N})),$$

is an isometric isomorphism between the Banach spaces $\mathrm{ca}(\mathcal{P}(\mathbb{N}))$ and l_1.

For $\mu \in \mathrm{ba}(\mathcal{P}(\mathbb{N}))$ the formula

$$x_\mu^*(x) = \int x \cdot \mathrm{d}\mu, \quad x \in l_\infty$$

defines a continuous linear functional x_μ^* on l_∞ with $\|x_\mu^*\| = \|\mu\|$. We shall denote this functional simply by μ. If $x \in c_0$ then the sequence $x_n = (x(1), \ldots, x(n), 0, \ldots)$, $n \in \mathbb{N}$, converges uniformly to x so that, by the definition of the integral with respect to μ,

$$\mu(x) = \lim_{n\to\infty} \mu(x_n) = \lim_{n\to\infty} \sum_{i=1}^n \mu(\{i\})x(i)$$

i.e.

$$\mu(x) = \sum_{i=1}^\infty \mu(\{i\})x(i), \quad \forall x \in c_0. \tag{26}$$

A charge $\mu \in \mathrm{ba}(\mathcal{P}(\mathbb{N}))$ is called *purely finitely additive* (or a *pure charge*) if $0 \le \lambda \le |\mu|$ implies $\lambda = 0$, for every $\lambda \in \mathrm{ca}(\mathcal{P}(\mathbb{N}))$. Denote by $\mathrm{pba}(\mathcal{P}(\mathbb{N}))$ the set of all purely finitely additive measures. By Yosida-Hewitt theorem (see [43] or [2, p. 240]), every charge $\mu \in \mathrm{ba}(\mathcal{P}(\mathbb{N}))$ admits a unique decomposition, called *Yosida-Hewitt decomposition,* of the form

$$\mu = \mu_c + \mu_p \tag{27}$$

with μ_c countably additive and μ_p purely finitely additive.

In our case

$$\mu_c(A) = \sum_{i\in A} \mu(\{i\}) \tag{28}$$

and

$$\mu_p(A) = \mu(A) - \sum_{i\in A} \mu(\{i\}) \tag{29}$$

for every $A \subset \mathbb{N}$. It follows that $\mu_p(A) = 0$ for every finite subset A of $\mathbb{N}$. In particular $\mu(\{i\}) = 0, \quad i \in \mathbb{N}$. so that, by (26), $\mu_p(x) = 0, \quad x \in c_0$, showing that

$$\mathrm{pba}(\mathcal{P}(\mathbb{N})) \subset c_0^{\perp}.$$

Conversely, if $\mu \in c_0^{\perp}$ then, by (26),

$$\sum_{i=1}^{\infty} \mu(\{i\})x(i) = \mu(x) = 0, \quad \forall x \in c_0,$$

implying $\mu(\{i\}) = 0$, $i \in \mathbb{N}$, which by (28) yields $\mu_c = 0$ and $\mu = \mu_p \in \mathrm{pba}(\mathcal{P}(\mathbb{N}))$, i.e. $c_0^{\perp} \subset \mathrm{pba}\ (\mathcal{P}(\mathbb{N}))$.

Consequently

$$c_0^{\perp} = \mathrm{pba}(\mathcal{P}(\mathbb{N})). \tag{30}$$

If $\mu = \mu_c + \mu_p$ then, by (28),

$$\|\mu_c\| = \sum_{i=1}^{\infty} |\mu(\{i\})|.$$

By (29)

$$|\mu_p(A_1)| + \ldots + |\mu_p(A_k)| = |\mu(A_1)| + \ldots + |\mu_p(A_k)| - \sum_{i=1}^{\infty} |\mu(\{i\})|$$

for every decomposition $\mathbb{N} = A_1 \cup \ldots \cup A_k$ of $\mathbb{N}$ into pairwise disjoint sets. It follows $\|\mu_p\| = \|\mu\| - \|\mu_c\|$ or, equivalently,

$$\|\mu\| = \|\mu_c\| + \|\mu_p\|. \tag{31}$$

Consequently

$$\mathrm{ba}(\mathcal{P}(\mathbb{N})) = \mathrm{ca}(\mathcal{P}(\mathbb{N})) \oplus \mathrm{pba}(\mathcal{P}(\mathbb{N})) = \mathrm{ca}(\mathcal{P}(\mathbb{N})) \oplus c_0^{\perp}. \tag{32}$$

A closed subspace Y of a Banach space X is called an M-*ideal* if X^* can be decomposed into a direct sum

$$X^* = Y^{\perp} \oplus \hat{Y}$$

with $\|x^*\| = \|y^*\| + \|z^*\|$, for every $(x^*, y^*, z^*) \in X^* \times Y^{\perp} \times \hat{Y}$, such that $x^* = y^* + z^*$. M-ideals were considered first by E. Alfsen and Efros [1]. For a thorough exposition of the present day situation in M-ideals theory we recommend the monograph [17] (see also [35]). In this language, the relations (31) and (32) tell us that c_0 is an M-ideal in l_∞. From the general theory of M-ideals it follows that the space c_0 is proximinal in l_∞ and that every continuous linear functional on c_0 has a unique norm-preserving extension to l_∞. Since $\mu_p|_{c_0} = 0$, for $\mu \in$ba$(\mathcal{P}(\mathbb{N}))$, the formulae (16) become

$$d(\mu, c_0^\perp) = \|\mu|_{c_0}\| = \|\mu_c\| = \sum_{i=1}^{\infty} |\mu(\{i\})| \quad \text{and} \quad P_{c_0^\perp}(\mu) = \{\mu_p\}.$$

This is another way to obtain the results in Proposition 2.

Since c_0 is proximinal in l_∞ it follows that every $x \in l_\infty$ admits a Helly extension with respect to c_0. To obtain concrete representations in this situation we have to work with the bidual $l_\infty^{**} = \mathrm{ba}^*(\mathcal{P}(\mathbb{N}))$. But, as it is asserted in [2, p. 231], no satisfactory representation for the elements of $\mathrm{ba}^*(\mathcal{P}(\mathbb{N}))$ is known.

2.4. *Tietze extensions*

Let T be a locally compact Hausdorff topological space and S a nonvoid closed subset of T. Denote by $C_0(T)$ $(C_0(S))$ the space of all real-valued continuous functions on T (respectively S) vanishing at infinity. Equipped with the sup-norms $C_0(T)$ and $C_0(S)$ are Banach algebras and

$$Z(S) = \{F \in C_0(T) : F|_S = 0\}$$

is a closed ideal in $C_0(T)$. By Tietze extension theorem (see [14]) every $f \in C_0(S)$ admits a norm-preserving extension $F \in C_0(T)$. It follows that Proposition 1 can be applied to deduce that $Z(S)$ is a proximinal subspace of $C_0(T)$ and that the formulae

$$d(F, Z(S)) = \|F|_S\| \quad \text{and} \quad P_{Z(S)}(F) = F - E(F|_S) \tag{33}$$

hold.

Results of this kind have been obtained in [10, 37].

Using Dugundji's [12] vector-version of Tietze extension theorem one obtains the validity of the formulae (33) for $C_0(T, E)$, $C_0(S, E)$, with E a Banach space.

2.5. *Spaces of Lipschitz and Hölder functions*

Let (X, d) be a metric space. A function $F : X \to \mathbb{R}$ is called *Lipschitz* if

$$\|F\| := \sup\left\{\frac{|F(x_1) - F(x_2)|}{d(x_1, x_2)} : x_1, x_2 \in X,\ x_1 \neq x_2\right\} < \infty. \tag{34}$$

The quantity $\|F\|$ is called the *Lipschitz norm* of the function F and the space of all real-valued Lipschitz functions on X is denoted by $\mathrm{Lip}(X)$. Since $\|F\| = 0$ for constant functions, (34) is in fact only a semi-norm on $\mathrm{Lip}(X)$. There are several ways to transform $\mathrm{Lip}(X)$ into a Banach space. One consists in fixing a point $x_0 \in X$ and consider the space $\mathrm{Lip}_0(X)$ of all functions in $\mathrm{Lip}(X)$ vanishing at x_0. Other way consists in considering the space $\mathrm{BLip}(X)$

of all bounded functions in $\mathrm{Lip}(X)$, normed by

(35) $$\|F\|_1 = \|F\| + \|F\|_\infty$$

or by

(36) $$\|F\|_2 = \max\{\|F\|, \|F\|_\infty\},$$

where $\|F\|$ is given by (34) and $\|F\|_\infty$ is the sup-norm.

McShane [25] proved that every Lipschitz functions f, defined on a subset Y of X, admits a norm preserving extension to X (see also [8, 26]). The case $X = \mathbb{R}^n$ was considered by M. Kirszbraun [22]. Based on this result one can show that the space $\mathrm{Lip}_0(Y)$ has the extension property in $\mathrm{Lip}_0(X)$ so that, by Proposition 1,

$$d(F, Y^\perp) = \|f|_Y\| \quad \text{and} \quad P_{Y^\perp}(F) = F - E(F|_Y)$$

for every $F \in \mathrm{Lip}_0(X)$, where

$$Y^\perp = \{F \in \mathrm{Lip}_0(X) : F|_Y = 0\}.$$

Here Y is a subset of X containing the fixed point x_0.

Similar results hold for the spaces $\mathrm{BLip}(Y)$ and $\mathrm{BLip}(X)$ with respect to the norms (35) or (36) (see [29]). In this case one can show that every $f \in \mathrm{BLip}(Y)$ has an extension $F \in \mathrm{BLip}(X)$ preserving both the Lipschitz and uniform norms, implying that the space $\mathrm{BLip}(Y)$ have the extension property with respect to $\mathrm{BLip}(X)$ for both of the norms (35) and (36).

For $0 < \alpha \leq 1$, a function $F : X \to \mathbb{R}$ is called *Hölder of order* α if

$$\|F\|_\alpha := \sup\left\{\frac{|F(x_1) - F(x_2)|}{d^\alpha(x_1, x_2)} : x_1, x_2 \in X,\ x_1 \neq x_2\right\} < \infty.$$

Denote by $\Lambda^\alpha(X)$ the space of all Hölder functions on X. To obtain Banach spaces of Hölder functions one can proceed like above, by considering the space $\Lambda_0^\alpha(X)$ of Hölder functions vanishing at a fixed point x_0 or the space $\mathrm{B}\Lambda^\alpha(X)$ of bounded Hölder functions on X. Duality results for these spaces have been obtained by C. Mustăţa [31, 32].

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