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HOMOGENEOUS NUMERICAL CUBATURE FORMULAS OF INTERPOLATORY TYPE

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Abstract. In this paper we construct homogeneous numerical cubature formulas based on some numerical multivariate interpolation schemes.

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1. INTRODUCTION

Let D be a given domain in $\mathbb{R}^2$, $f : D \to \mathbb{R}$ an integrable function on D and $\Lambda := \{\lambda_1 f, \dots, \lambda_N f\}$ some given information on f. Next, one suppose that $\lambda_i f$ are values of f or of certain of its derivatives at some points of D, called nodes.

One considers the cubature formula

$$I^{xy} f := \iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{N} A_i \lambda_i f + R_N(f),$$

where A_i, $i = 1, \dots, N$ are its coefficients and $R_N(f)$ is the remainder term.

The coming problem is to find the parameters of such a cubature formula (coefficients, nodes) and to study the remainder term.

The most results has been obtained when D is a regular domain in $\mathbb{R}^2$ (rectangle, triangle) and the information (data) are regularly spaced. At this class of cubature procedure belong the tensorial product and the cubature sum rules.

Let $D \in \mathbb{R}^2$ be a rectangle, $D = [a,b] \times [c,d]$.

If $\Lambda^x := \{\lambda_i^x f \mid i = 0, 1, \dots, m\}$ and $\Lambda^y := \{\lambda_j^y f \mid j = 0, 1, \dots, n\}$, $m, n \in \mathbb{N}$ are given sets of information on f with regard to x respectively y, one considers the quadrature formulas

$$I^x f := \int_a^b f(x,y)\mathrm{d}x = (Q_1^x f)(\cdot, y) + (R_1^x f)(\cdot, y)$$

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and

$$I^y f := \int_c^d f(x,y)\mathrm{d}y = (Q_1^y f)(x,\cdot) + (R_1^y f)(x,\cdot),$$

where the quadrature rules Q_1^x and Q_1^y are given by

$$(Q_1^x f)(\cdot, y) = \sum_{i=0}^{m} A_i(\lambda_i^x f)(\cdot, y),$$

respectively

$$(Q_1^y f)(x, \cdot) = \sum_{j=0}^{n} B_j(\lambda_j^y f)(x, \cdot),$$

with R_1^x and R_1^y the corresponding remainder operators, i.e. $R_1^x = I^x - Q_1^x$, $R_1^y = I^y - Q_1^y$.

It is easy to check the following decomposition of the double integral operator I^{xy}

$$I^{xy} = Q_1^x Q_1^y + (R_1^x I^y + I^x R_1^y - R_1^x R_1^y) \tag{1}$$

and

$$I^{xy} = (Q_1^x I^y + I^x Q_1^y - Q_1^x Q_1^y) + R_1^x R_1^y. \tag{2}$$

The identities (1) and (2) generate so called product cubature formula

$$I^{xy} f = Q_1^x Q_1^y f + (R_1^x I^y + I^x R_1^y - R_1^x R_1^y) f, \tag{3}$$

respectively the boolean-sum cubature formula

$$I^{xy} f = (Q_1^x I^y + I^x Q_1^y - Q_1^x Q_1^y) f + R_1^x R_1^y f. \tag{4}$$

Let p_1 and q_1 be the approximation order of Q_1^x, respectively Q_1^y: $\mathrm{ord}(Q_1^x) = p_1$, $\mathrm{ord}(Q_1^y) = q_1$ [4].

From (3) and (4) it follows that the approximation order of the product formula is $\min\{p_1, q_1\}$ while the approximation order of the boolean-sum formula is $p_1 + q_1$.

Hence, the boolean-sum cubature rules has the remarkable property regarding its highest approximation order.

Otherwise, the boolean-sum formula contains the simple integrals $I^x f$, respectively $I^y f$. But, this simple integrals can be approximated, in a second level of approximation, using new quadrature procedures.

From (4), one obtains

$$I^{xy} f = Qf + Rf \tag{5}$$

with $Q = Q_1^x Q_2^y + Q_2^x Q_1^y - Q_1^x Q_1^y$ and $R = Q_1^x R_2^y + Q_1^y R_2^x + R_1^x R_1^y$, where Q_2^x and Q_2^y are the quadrature rules used in the second level of approximation and R_2^x, R_2^y are the corresponding remainder operators.

As can be seen

$$\mathrm{ord}(Q) = \min\{\mathrm{ord}(Q_1^x) + \mathrm{ord}(Q_1^y),\ \mathrm{ord}(Q_2^x) + 1,\ \mathrm{ord}(Q_2^y) + 1\}.$$

The quadrature rules Q_2^x and Q_2^y can be chosen in many ways. First of all, it depends on the given information of the function f.

A natural way to choose them is such that the approximation order of the initial boolean-sum formula to be preserved. It is obvious that its approximation order cannot be increased.

Definition 1. *A cubature formula of the form* (5) *derived from the boolean-sum formula* (4) *which preserves its approximation order is called a consistent cubature formula.*

Remark 1. The cubature formula (5) is consistent if the orders p_2 and q_2 of the quadrature procedures Q_2^x, respectively Q_2^y, used in the second level of approximation, satisfy the inequalities $p_2 \geq p_1 + q_1 - 1$, $q_2 \geq p_1 + q_1 - 1$. $\square$

As the approximation order of the boolean-sum cubature cannot be increased, it is preferable to choose the quadrature procedures Q_2^x and Q_2^y such that each term of the remainder from (5) to have the same order of approximation.

Definition 2. *A cubature formula, of the form* (5)*, of which each term of the remainder has the same order of approximation is called a homogeneous cubature formula.*

Remark 2. The cubature formula (5) is homogeneous if $p_2 = q_2 = p_1 + q_1 - 1$.

For example, let be

$$I^{xy}f = \int_0^h \int_0^h f(x,y)\mathrm{d}x\mathrm{d}y,$$

and

$$(Q_1^x f)(\cdot, y) = hf\left(\tfrac{h}{2}, y\right), \quad \text{respectively } (Q_1^y f)(x, \cdot) = hf\left(x, \tfrac{h}{2}\right)$$

the gaussian quadrature rules. Then for boolean-sum cubature formula, we have

$$I^{xy}f = h\int_0^h f\left(\tfrac{h}{2}, y\right)\mathrm{d}y + h\int_0^h f\left(x, \tfrac{h}{2}\right)\mathrm{d}x - h^2 f\left(\tfrac{h}{2}, \tfrac{h}{2}\right) + R_S(f),$$

where

$$R_S(f) = \tfrac{h^6}{576} f^{(2,2)}(\xi, \eta). \qquad \square$$

In order to get a homogeneous numerical cubature formula we must use, in a second level of approximation, some quadrature rules Q_2^x and Q_2^y with $\mathrm{ord}(Q_2^x) = \mathrm{ord}(Q_2^y) = 5$. Such quadrature rules can be

$$(Q_2^x f)(\cdot, y) = \tfrac{h}{2}[f(0,y) + f(h,y)] + \tfrac{h^2}{12}[f^{(1,0)}(0,y) - f^{(1,0)}(h,y)]$$

with

$$(R_2^x f)(\cdot, y) = \tfrac{h^5}{720} f^{(4,0)}(\xi_1, y),$$

respectively

$$(Q_2^y f)(x,\cdot) = \tfrac{h}{2}[f(x,0)+f(x,h)] + \tfrac{h^2}{12}[f^{(0,1)}(x,0) - f^{(0,1)}(x,h)]$$

with

$$(R_2^y f)(x,\cdot) = \tfrac{h^5}{720} f^{(0,4)}(x,\eta_1).$$

It follows:

THEOREM 1. *If* $f^{(4,0)}\left(\cdot,\frac{h}{2}\right), f^{(0,4)}\left(\frac{h}{2},\cdot\right) \in C[0,h]$ *and* $f^{(2,2)} \in C(D_h)$, *with* $D_h = [0,h]\times[0,h]$, *then we have the homogeneous cubature formula*

$$\begin{aligned}(6)\quad \iint_{D_h} f(x,y)\mathrm{d}x\mathrm{d}y &= \\ &= \tfrac{h^2}{2}\left[f\left(\tfrac{h}{2},0\right) + f\left(\tfrac{h}{2},h\right) + f\left(0,\tfrac{h}{2}\right) + f\left(h,\tfrac{h}{2}\right) - 2f\left(\tfrac{h}{2},\tfrac{h}{2}\right)\right] \\ &\quad + \tfrac{h^3}{12}\left[f^{(1,0)}\left(0,\tfrac{h}{2}\right) - f^{(1,0)}\left(h,\tfrac{h}{2}\right) + f^{(0,1)}\left(\tfrac{h}{2},0\right) - f^{(0,1)}\left(\tfrac{h}{2},h\right)\right] + R(f),\end{aligned}$$

where

$$R(f) = \tfrac{h^6}{144}\left[\tfrac{1}{5}f^{(4,0)}\left(\tfrac{h}{2},\eta_1\right) + \tfrac{1}{5}f^{(0,4)}\left(\xi,\tfrac{h}{2}\right) + \tfrac{1}{4}f^{(2,2)}(\xi,\eta)\right]. \tag{7}$$

Next, we try to construct such homogeneous cubature formulas using interpolation formulas derived from a boolean-sum interpolation formula and not only.

It is know [2] that if P_1^x and P_1^y are univariate interpolation operators, from the corresponding boolean-sum formula

$$f = P_1^x \oplus P_1^y f + R_1^x R_1^y f$$

can be derived using in a second level of approximation, some new operators P_2^x and P_2^y, a numerical approximation formula, i.e.

$$f = (P_1^x P_2^y + P_2^x P_1^y - P_1^x P_1^y)f + (P_1^x R_2^y + P_1^y R_2^x + R_1^x R_1^y)f. \tag{8}$$

Now, if f is an integrable function on $D=[a,b]\times[c,d]$ then it is obviously that

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \iint_D (P_1^x P_2^y + P_2^x P_1^y - P_1^x P_1^y)f(x,y)\mathrm{d}x\mathrm{d}y + R(f),$$

where

$$R(f) = \iint_D ((P_1^x R_2^y + P_1^y R_2^x + R_1^x R_1^y)f)(x,y)\mathrm{d}x\mathrm{d}y$$

is a numerical integration formula.

For $f : D_h \to \mathbb{R}$, one considers

$$\Lambda^x = \left\{ f\left(\tfrac{h}{m}i,\cdot\right) \mid i=0,1,\dots,m\right\},\quad \Lambda^y = \left\{ f\left(\cdot,\tfrac{h}{n}j\right) \mid j=0,1,\dots,n\right\}$$

$S_1^x f$ and $S_3^x f$ the linear respectively the cubic spline that interpolates f with regard to Λ^x and $S_1^y f, S_3^y f$ the corresponding splines that interpolates f with regard to Λ^y, i.e.

$$(S_1^x f)(x,\cdot) = \sum_{i=0}^{m} s_i^1(x) f\left(\tfrac{h}{m}i,\cdot\right), \quad (S_3^x f)(x,\cdot) = \sum_{i=0}^{m} s_i^3(x) f\left(\tfrac{h}{m}i,\cdot\right),$$

$$(S_1^y f)(\cdot,y) = \sum_{j=0}^{n} s_j^1(y) f\left(\cdot,\tfrac{h}{n}j\right), \quad (S_3^y f)(\cdot,y) = \sum_{j=0}^{n} s_j^3(y) f\left(\cdot,\tfrac{h}{n}j\right),$$

where $s_i^1, s_i^3, s_j^1, s_j^3$ are the corresponding cardinal splines.

Theorem 2. *Let f be an integrable function on D_h.*

$$\iint\limits_{D_h} f(x,y)\mathrm{d}x\mathrm{d}y = \sum_{i=0}^{m}\sum_{j=0}^{n}(A_i^1 B_j^3 + A_i^3 B_j^1 - A_i^1 B_j^1) f\left(\tfrac{h}{m}i, \tfrac{h}{n}j\right) + R_{mn}(f)$$

where

$$A_i^1 = \int_0^h s_i^1(x)\mathrm{d}x,\;\; A_i^3 = \int_0^h s_i^3(x)\mathrm{d}x,\;\; B_j^1 = \int_0^h s_j^1(y)\mathrm{d}y,\;\; B_j^3 = \int_0^3 s_j^3(y)\mathrm{d}y$$

is a homogeneous cubature formula.

Proof. The bivariate interpolation formula (8) becomes $f = S_{13}^{xy} f + R_{mn} f$ with $S_{13}^{xy} = S_1^x S_3^y + S_3^x S_1^y - S_1^x S_1^y$ and $R_{mn} = S_1^x R_3^y + S_1^y R_3^x + R_1^x R_1^y$, or

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n}[s_i^1(x)s_j^3(y) + s_i^3(x)s_j^1(y) - s_i^1(x)s_j^1(y)] f\left(\tfrac{h}{m}i, \tfrac{h}{n}j\right) + (R_{mn}f)(x,y).$$

Now,

$$\iint\limits_{D_h} f(x,y)\mathrm{d}x\mathrm{d}y = (Q_1^x Q_3^y + Q_3^x Q_1^y - Q_1^x Q_1^y) f + R_{mn}(f) \tag{9}$$

with

$$(Q_1^x f)(\cdot,y) = \sum_{i=0}^{m} A_i^1 f\left(\tfrac{h}{m}i, y\right), \quad (Q_1^y f)(x,\cdot) = \sum_{j=0}^{n} B_j^1 f\left(x, \tfrac{h}{n}j\right),$$

$$(Q_3^x f)(\cdot,y) = \sum_{i=0}^{m} A_i^3 f\left(\tfrac{h}{m}i, y\right), \quad (Q_3^y f)(x,\cdot) = \sum_{j=0}^{n} B_j^3 f\left(x, \tfrac{h}{n}j\right),$$

where

$$R_{mn}(f) = \iint\limits_{D_h}(S_1^x R_3^y + S_1^y R_3^x + R_1^x R_1^y) f(x,y)\mathrm{d}x\mathrm{d}y.$$

But in general the degree of exactness ("dex") of the linear spline operators are: $\mathrm{dex}(S_1^x) = \mathrm{dex}(S_1^y) = 0$ and $\mathrm{dex}(S_3^x) = \mathrm{dex}(S_3^y) = 1$, so $\mathrm{dex}(Q_1^x) = \mathrm{dex}(Q_1^y) = 0$, $\mathrm{dex}(Q_3^x) = \mathrm{dex}(Q_3^y) = 1$. It follows that [3] $\mathrm{ord}(Q_1^x) = \mathrm{ord}(Q_1^y) = 2$, $\mathrm{ord}(Q_3^x) = \mathrm{ord}(Q_3^y) = 3$. Hence

$$\mathrm{ord}(Q_3^x) = \mathrm{ord}(Q_3^y) = \mathrm{ord}(Q_1^x) + \mathrm{ord}(Q_1^y) - 1. \qquad \square$$

Remark 3. Taking into account that if

$$f(x) = (Sf)(x) + (Rf)(x)$$

is a natural spline interpolation formula then

$$\int_a^b f(x)dx = \int_a^b (Sf)(x)dx + \int_a^b (Rf)(x)\mathrm{d}x$$

is an optimal quadrature formula in sense of Sard [7], formula (9) is also an almost optimal cubature formula [9]. □

Some particular cases

(C1) Let S_1^x and S_1^y be the Lagrange-type linear spline operators that interpolate the data

$$\Lambda_1^x = \left\{f(0,y), f\left(\tfrac{h}{2}, y\right), f(h,y)\right\}, \quad \text{resp.} \quad \Lambda_1^y = \left\{f(x,0), f\left(x, \tfrac{h}{2}\right), f(x,h)\right\}$$

and S_3^x and S_3^y the Hermite-type cubic spline operators that interpolate

$$\Lambda_3^x = \left\{f(0,y), f^{(1,0)}(0,y), f\left(\tfrac{h}{2}, y\right), f(h,y), f^{(1,0)}(h,y)\right\},$$

respectively

$$\Lambda_3^y = \left\{f(x,0), f^{(0,1)}(x,0), f\left(x, \tfrac{h}{2}\right), f(x,h), f^{(0,1)}(x,h)\right\}.$$

Let also

$$f = (S_1^x S_3^y + S_3^x S_1^y - S_1^x S_1^y)f + (S_1^y R_3^x + S_1^x R_3^y + R_1^x R_1^y)f$$

be the spline interpolation formula generated by the operators S_1^x, S_1^y, S_3^x and S_3^y using two level of approximation with $R_1^x, R_1^y, R_3^x, R_3^y$ the corresponding remainder operators.

Theorem 3. *If $f \in C^{(4,4)}(D_h)$ then*

$$\iint\limits_{D_h} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{D_h} (S_1^x S_3^y + S_3^y S_1^y - S_1^x S_1^y)f(x,y)\mathrm{d}x\mathrm{d}y + (R_{13}f)(x,y),$$

where

$$(R_{13}f)(x,y) = \iint\limits_{D_h} (S_1^y R_3^x + S_1^x R_3^y + R_1^x R_1^y)f(x,y)\mathrm{d}x\mathrm{d}y$$

is a homogeneous cubature formula of order 6.

Proof. We must check that $\mathrm{dex}(S_1^x) = 1$ and $\mathrm{dex}(S_3^x) = 3$. By the symmetry of the data it follows that also $\mathrm{dex}(S_1^y) = 1$ and $\mathrm{dex}(S_3^y) = 3$.

We have

$$(S_1^x f)(x,y) = s_0(x)f(0,y) + s_1(x)f\left(\tfrac{h}{2}, y\right) + s_2(x)f(h,y)$$

with

$$s_0(x) = 1 - \tfrac{2}{h}x + \tfrac{2}{h}\left(x - \tfrac{h}{2}\right)_+, \quad s_1(x) = \tfrac{2}{h}x - \tfrac{4}{h}\left(x - \tfrac{h}{2}\right)_+, \quad s_2(x) = \tfrac{2}{h}\left(x - \tfrac{h}{2}\right)_+.$$

Now it is easy to verify that $S_1^x e_i = e_i$ for $i = 0,1$ and $S_1^x e_2 = e_2$, with $e_i(x) = x^i$.

We also have

$$\begin{aligned}(S_3^x f)(x,y) &= s_{00}(x) f(0,y) + s_{01}(x) f^{(1,0)}(0,y) + s_{10}(x) f\left(\tfrac{h}{2}, y\right)\\ &\quad + s_{20}(x) f(h,y) + s_{21}(x) f^{(1,0)} f(h,y)\end{aligned}$$

with

$$\begin{aligned}s_{00}(x) &= 1 + \tfrac{10}{h^3}x^3 - \tfrac{9}{h^2}x^2 - \tfrac{16}{h^3}\left(x - \tfrac{h}{2}\right)_+^3,\\ s_{01}(x) &= x + \tfrac{3}{h^2}x^3 - \tfrac{7}{2h}x^2 - \tfrac{4}{h^2}\left(x - \tfrac{h}{2}\right)_+^3\\ s_{10}(x) &= -\tfrac{16}{h^3}x^3 + \tfrac{12}{h^2}x^2 + \tfrac{32}{h^3}\left(x - \tfrac{h}{2}\right)_+^3,\\ s_{20}(x) &= \tfrac{6}{h^3}x^3 - \tfrac{3}{h^2}x^2 - \tfrac{16}{h^3}\left(x - \tfrac{h}{2}\right)_+^3\\ s_{21}(x) &= -\tfrac{1}{h^2}x^3 + \tfrac{1}{2h}x^2 + \tfrac{4}{h^2}\left(x - \tfrac{h}{2}\right)_+^3.\end{aligned}$$

But $S_3^\eta e_i = e_i$ for $i = 0,1,2,3$. □

Remark 4. The corresponding quadrature rules are

$$(Q_1^x f)(\cdot, y) := \int_0^h (S_1^x f)(x,y)\mathrm{d}x = \tfrac{h}{4}\left[f(0,y) + 2f\left(\tfrac{h}{2}, y\right) + f(h,y)\right],$$

with

$$(R_1^x f)(\cdot, y) = -\frac{h^3}{48} f^{(2,0)}(\xi_1, y)$$

and

$$\begin{aligned}(Q_3^x f)(\cdot, y) &:= \int_0^h (S_3^x f)(x,y)\mathrm{d}x\\ &= \tfrac{h}{4}\left[f(0,y) + \tfrac{h}{12} f^{(1,0)}(0,y) + 2f\left(\tfrac{h}{2}, y\right) + f(h,y) - \tfrac{h}{12} f^{(1,0)}(h,y)\right],\end{aligned}$$

where

$$(R_3^x f)(\cdot, y) = \tfrac{h^5}{11520} f^{(4,0)}(\xi_2, y).$$ □

(C2) If in the particular case (C1) is taken instead of the Hermite-type cubic spline operators the next Birkhoff-type cubic spline operators S_3^x and S_3^y that interpolate the date

$$\Lambda_3^x = \left\{f^{(1,0)}(0,y) + f\left(\tfrac{h}{2}, y\right) + f^{(1,0)}(h,y)\right\},$$

respectively

$$\Lambda_3^y = \left\{f^{(0,1)}(x,0), f\left(x, \tfrac{h}{2}\right), f^{(0,1)}(x,h)\right\}$$

then we obtain a homogeneous cubature formula of the order 6. Certainly, this is the case if $\mathrm{dex}(Q_3^x) = \mathrm{dex}(Q_3^y) = 3$, where

$$(Q_3^x f)(\cdot, y) = \iint_{D_h} (S_3^x f)(x, y)\mathrm{d}x \quad \text{and} \quad (Q_3^y f)(x, \cdot) = \iint_{D_h} (S_3^y)(x, y)\mathrm{d}y.$$

But

$$(S_3^x f)(x, y) = s_{01}(x) f^{(1,0)}(0, y) + s_{10}(x) f\left(\tfrac{h}{2}, y\right) + s_{21}(x) f^{(1,0)}(h, y),$$

where

$$s_{01}(x) = -\tfrac{3h}{8} + x - \tfrac{1}{2h}x^2, \qquad s_{10}(x) = 1, \qquad s_{21}(x) = -\tfrac{h}{8} + \tfrac{1}{2h}x^2.$$

By a straightforward computation one obtains that

$$S_3^x = e_i, \quad \text{for } i = 0, 1, 2,$$

i.e. $\mathrm{dex}(S_3^x) = 2$. Hence $\mathrm{dex}(Q_3^x) \geq 2$. Taking into account that

$$(Q_3^x f)(\cdot, y) = -\tfrac{h^2}{24} f^{(1,0)}(0, y) + hf\left(\tfrac{h}{2}, y\right) + \tfrac{h^2}{24} f^{(1,0)}(h, y)$$

it follows that $\mathrm{dex}(Q_3^x) = 3$.

(C3) Let f be an integrable function on the standard triangle

$$T_h := \{(x, y) \in \mathbb{R}^2 |\ x \geq 0,\ y \geq 0,\ x + y \leq h\}.$$

For $f \in B_{1,2}(0, 0)$ [6], one considers the Birkhoff-type bivariate polynomial operator say B_2 that interpolates the date

$$\Lambda_B = \{f(0, 0) + f^{(2,0)}(0, 0), f^{(1,1)}(0, 0), f^{(0,2)}(0, 0), f(h, 0), f(0, h)\},$$

i.e.

$$\begin{aligned}(B_2 f)(x, y) = {} & \tfrac{h-x-y}{h} f(0, 0) + \tfrac{x(x-h)}{2} f^{(2,0)}(0, 0) + xyf^{(1,1)}(0, 0) \\ & + \tfrac{y(y-h)}{2} f^{(0,2)} f(0, 0) + \tfrac{x}{h} f(h, 0) + \tfrac{y}{h} f(0, h).\end{aligned}$$

It follows that

$$\begin{aligned}Q_2(f) = {} & \tfrac{h^2}{6} [f(0, 0) - \tfrac{h^2}{4} f^{(2,0)}(0, 0) + \tfrac{h^2}{4} f^{(1,1)}(0, 0) \\ & - \tfrac{h^2}{4} f^{(0,2)}(0, 0) + f(h, 0) + f(0, h)].\end{aligned}$$

It is easy to check that $\mathrm{dex}(Q_2) = 2$. In other words $Q_2(f) = f$ for all $f \in \mathbb{P}_2^2$ (the set of the bivariate polynomials of the total degree at most 2).

Now, let us consider the cubature formula

$$\iint_{T_h} f(x, y)\mathrm{d}x\mathrm{d}y = Q_2(f) + R_2(f). \tag{10}$$

Using the Peano-type theorem for bivariate case [1], one obtains

$$R_2(f) = \tfrac{h^5}{120} \left[-\tfrac{7}{3} f^{(3,0)}(\xi, 0) + f^{(2,1)}(\xi_1, 0) - \tfrac{7}{3} f^{(0,3)}(0, \eta) - f^{(1,2)}(\xi_2, \eta_1)\right].$$

So, the cubature formula (10) is a homogeneous one, of order 5.

REFERENCES

[1] Barnhill, R. E. and Mansfield, L., *Error bounds for smooth interpolation in triangles*, J. Approx. Theory, **11**, pp. 306–318, 1974.

[2] Coman, Gh., *Multivariate approximation schemes and the approximation of linear functions*, Mathematica, **16** (**39**), no. 2, pp. 229–249, 1974.

[3] Coman, Gh., *The complexity of the quadrature formulas*, Mathematica (Cluj), **23** (**46**), pp. 183–192, 1981.

[4] Coman, Gh., *Homogeneous cubature formulas*, Studia Univ. Babeş-Bolyai, Mathematica, **38**, no. 2, pp. 91–101, 1993.

[5] Delvos, F. F., *Boolean methods for double integration*, Math. Comp., **55**, pp. 683–692, 1990.

[6] Sard, A., *Linear Approximation*, Amer. Math. Society, Providence, 1963.

[7] Schoenberg, I. J., *On best approximation of linear operators*, Indag. Math., **26**, pp. 155–163, 1964.

[8] Smolyak, S. A., *Quadrature and interpolation formulas for tensor products of certain classes of functions*, Soviet Math. Dokl., **4**, pp. 240–243, 1963.

[9] Somogyi, I., *Almost optimal numerical methods*, Studia Univ. Babeş-Bolyai, Mathematica, **1**, pp. 85–93, 1999.

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