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STANCU CURVES IN CAGD

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Abstract. Starting from the one-parameter dependent linear polynomial Stancu operator, we consider the related polynomial curve scheme with one scalar shape parameter. This scheme, called by us the Stancu curve scheme, generalizes in a suitable manner the classical Bernstein-Bézier scheme and provides more design flexibility by means of the shape parameter.

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1. INTRODUCTION

Let us begin by recalling that the (one-parameter dependent linear polynomial) Stancu operator $S^{n;\alpha}$, for the interval $I = [0,1]$ and a function $f : I \to \mathbb{R}$,

$$\left(S^{n;\alpha} f\right)(x) = \sum_{i=0}^{n} S_i^{n;\alpha}(x) f\left(\tfrac{i}{n}\right),$$

where α is a real parameter and

$$S_i^{n;\alpha}(x) = \binom{n}{i} \frac{\prod_{j=0}^{i-1}(x + j\alpha) \prod_{j=0}^{n-i-1}(1 - x + j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)},$$

has been introduced and investigated by D. D. Stancu already in 1968 (see [16] where the polynomials $S_i^{n;\alpha}(x)$ are denoted by $w_{n,i}(x;\alpha)$).

The Stancu operator was studied further by Stancu in his subsequent papers [17], [18], as well as in several papers published by other authors (see [8], [5], [14], and [15]).

Although the Stancu operator has many properties similar to the Bernstein operator, until now only a few considerations from the point of view of CAGD (computer aided geometric design) of the Stancu polynomials $S_i^{n;\alpha}$ has been made: [9], [11], [10], [7], [19], [20], [21].

Starting with the classical Pólya's urn model as introduced by F. Eggenberger and G. Pólya [4], R. N. Goldman [11] studies the corresponding one-parameter dependent probability distributions $D_i^n(a,x)$ and the curve scheme related to these distributions. The distributions $D_i^n(a,x)$ are exactly the

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Stancu polynomials $S_i^{n;\alpha}(x)$ as observed by Stancu in [17]. The curves that use the distributions $D_i^n(a,x)$ as blending functions are called by Goldman Pólya curves. Ph. J. Barry and R. N. Goldman generalize this one-parameter dependent curve scheme to a curve scheme depending on $2n$ parameter, where n is the degree of the curves. This generalized curves are also called Pólya curves. We will use the term Stancu curves for the one-parameter case, letting the term Pólya curves denote the multi-parameter case. Thus, for us a Stancu curve is a uniform Pólya curve, where the parameter α gives the step of the progressive knot sequence (for details see [2], [3] or Sec. 4 bellow).

The aim of this paper is to study the polynomial curve scheme based on the Stancu polynomials. We begin in Section 2 by giving a brief overview of factorial powers and generalized binomial coefficients. Then, in Section 3 we formally introduce the Stancu polynomials and prove the first properties. The Stancu curve scheme is presented in Section 4. In Section 5 we establish the degree elevation formula, while Section 6 is devoted to the derivative formula for the new curve scheme. Conclusions are presented in the last section.

2. FACTORIAL POWERS AND GENERALIZED BINOMIAL COEFFICIENTS

In this brief section we recall some useful facts concerning the factorial powers and the generalized binomial coefficients and establish our notation.

Let α be a fixed real number. For each $x \in \mathbb{R}$ and $n \in \mathbb{N}$ the n^{th} *factorial power of* x (*with respect to* α) is defined by

$$x^{[0;\alpha]} := 1, \quad x^{[n;\alpha]} := x^{[n-1;\alpha]} \cdot (x + (n-1)\alpha),\ n = 1, 2, ... \tag{2.1}$$

Although the factorial power depends explicitly on the value of the parameter α, when the context is clear, we shall write simply $x^{[n]}$ instead of $x^{[n;\alpha]}$.

From this definition follows immediately that

$$x^{[n]} = \prod_{i=0}^{n-1} (x + i\alpha) \tag{2.2}$$

for every $n \in \mathbb{N}$, where the void product is considered to be equal to 1.

For each $n \in \mathbb{N}$ we define

$$x^{[-n]} := \frac{1}{x^{[n]}}\ . \tag{2.3}$$

The following properties can be easily derived from the definition above:

$$x^{[m+n]} = x^{[m]} \cdot (x + m\alpha)^{[n]} = x^{[n]} \cdot (x + n\alpha)^{[m]}, \tag{2.4}$$

$$\frac{x^{[m]}}{x^{[n]}} = \big(x + \min\{m,n\}\big)^{[m-n]} \tag{2.5}$$

for $n, m \in \mathbb{N}$.

Now it is easy to see that for each $n \in \mathbb{N}$ and for two arbitrary real numbers a and b, the binomial formula generalizes to

$$(a+b)^{[n]} = \sum_{i=0}^{n} \tbinom{n}{i} a^{[i]} b^{[n-i]} . \tag{2.6}$$

To simplify some formulae we use the *generalized binomial coefficients* (*w*ith respect to α) $\left[{n \atop i}\right]_\alpha$ defined by the following recurrence:

$$\left[{n \atop 0}\right]_\alpha := 1, \quad \left[{n \atop i}\right]_\alpha := \tfrac{1}{1+(n-1)\alpha} \left(\left[{n-1 \atop i}\right]_\alpha + \left[{n-1 \atop i-1}\right]_\alpha \right) . \tag{2.7}$$

Again, if the context is clear, we shall write simply $\left[{n \atop i}\right]$ instead of $\left[{n \atop i}\right]_\alpha$. From the definition follows immediately that

$$\left[{n \atop i}\right] = \frac{\binom{n}{i}}{1^{[n]}} . \tag{2.8}$$

Now, the generalized binomial formula can be written as

$$(a+b)^{[n]} = 1^{[n]} \sum_{i=0}^{n} \left[{n \atop i}\right] a^{[i]} b^{[n-i]} . \tag{2.9}$$

3. STANCU POLYNOMIALS

This section formally introduces the Stancu polynomials and gives the first properties.

For $x \in \mathbb{R}$ we write $u(x) = u = x$ and $v(x) = v = 1 - x$. Note that v and u are the affine coordinates of the point $x \in \mathbb{R}$ w.r.t. the interval $I = [0, 1]$. Consider $n \in \mathbb{N}$ and α a fixed real number such that $\alpha \in \left[-\frac{1}{n}, \infty\right)$ if $n \neq 0$, or $\alpha = 0$ if $n = 0$. For $x \in \mathbb{R}$, $r = 0, 1, \ldots, n$ and $i = 0, 1, \ldots, r$ we define the polynomials

$$\begin{aligned} (3.1)\ & S_0^{r;\alpha}(x) := S_0^0(x) \equiv 1, \qquad r = 0 \\ & S_i^{r;\alpha}(x) := \tfrac{u+(i-1)\alpha}{1+(r-1)\alpha} S_{i-1}^{r-1;\alpha}(x) + \tfrac{v+(r-i-1)\alpha}{1+(r-1)\alpha} S_i^{r-1;\alpha}(x), \quad r = 1, 2, \ldots, n , \end{aligned}$$

where $S_j^{s;\alpha}$ is considered null if $j \notin \{0, 1, \ldots, s\}$. Again, when the context is clear, we drop the superscript α and denote $S_i^{r;\alpha}$ simply by S_i^r. Clearly, S_i^r are univariate polynomials of degree at most n depending on the independent variable x. In fact, S_i^r are polynomials of degree exactly n, as we shall see later (see Remark 3.1). Now we give the formal definition of the Stancu polynomials and prove that they satisfy the recurrence relation (3.1).

DEFINITION 3.1. *Let $n \in \mathbb{N}$ and α be a fixed real number such that $\alpha \in \left[-\frac{1}{n}, \infty\right)$ if $n \neq 0$, or $\alpha = 0$ if $n = 0$. For $r = 0, 1, \ldots, n$ and $i = 0, 1, \ldots, r$ the polynomial*

$$\left[{r \atop i}\right] u^{[i]} v^{[r-i]} \tag{3.2}$$

(*where the factorial powers and the generalized binomial coefficient are considered w.r.t. the parameter* α) *is called the* i^{th} Stancu polynomial of degree r over the interval $[0,1]$ (with the parameter α).

Theorem 3.2. *The polynomials* S_i^r, $r = 0, 1, \ldots, n$, $i = 0, 1, \ldots, r$, *are the Stancu polynomials, namely*

$$S_i^r = \begin{bmatrix} r \\ i \end{bmatrix} u^{[i]} v^{[r-i]}. \tag{3.3}$$

Proof. We use induction on n. The result is certainly true for $r = 0$. Assume it is so for $r - 1$. Then by the recursion formula (3.1)

$$\begin{aligned} S_i^r &= \tfrac{u+(i-1)\alpha}{1+(r-1)\alpha} S_{i-1}^{r-1} + \tfrac{v+(r-i-1)\alpha}{1+(r-1)\alpha} S_i^{r-1} \\ &= \tfrac{u+(i-1)\alpha}{1+(r-1)\alpha} \begin{bmatrix} r-1 \\ i-1 \end{bmatrix} u^{[i-1]} v^{[r-i-1]} + \tfrac{v+(r-i-1)\alpha}{1+(r-1)\alpha} \begin{bmatrix} r-1 \\ i \end{bmatrix} u^{[i]} v^{[r-i-1]} \\ &= \tfrac{1}{1+(r-1)\alpha} \left(\begin{bmatrix} r-1 \\ i-1 \end{bmatrix} + \begin{bmatrix} r-1 \\ i \end{bmatrix} \right) u^{[i]} v^{[r-i]} \\ &= \begin{bmatrix} r \\ i \end{bmatrix} u^{[i]} v^{[r-i]}, \end{aligned}$$

where for the last equality we used (2.7). □

Remark 3.1. Similar to Bernstein polynomials, the Stancu polynomials have the following properties:

(1) The polynomials S_i^n are of degree n.

(2) For $\alpha = 0$, the Stancu polynomials specialize to the Bernstein polynomials:

$$S_i^n = B_i^n = \binom{n}{i} u^i v^{n-i}.$$

(3) For $\alpha = -\frac{1}{n}$, $n \neq 0$, the Stancu polynomials specialize to the Lagrange polynomials for the knots $x_i = \frac{i}{n}$, $i = 0, 1, \ldots, n$ on the interval $[0,1]$:

$$S_i^n = L_i^n = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x - x_j) \Bigg/ \prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j).$$

(4) For $\alpha \geq 0$, the Stancu polynomials are nonnegative over $[0,1]$.

(5) $S_i^n(0) = \delta_{i0}$ and $S_i^n(1) = \delta_{in}$ for $i = 0, 1, \ldots, n$. □

In the next section we present the curve scheme built on the Stancu polynomials. For a curve scheme the affine invariance property, i.e. the independence from the coordinate system is crucial. It is well known that this property is equivalent to the partition of unity property for the blending function of the scheme. The next theorem shows that the Stancu polynomials indeed partition the unity. We omit the straightforward proof by induction on n.

Theorem 3.3. *For* $\alpha \geq 0$ *the Stancu polynomials* S_i^n, $i = 0, 1, \ldots, n$, $n \in \mathbb{N}$ *form a partition of unity.*

Figure 3.1 illustrates the Stancu polynomials of degree 3 for different shape parameter values.

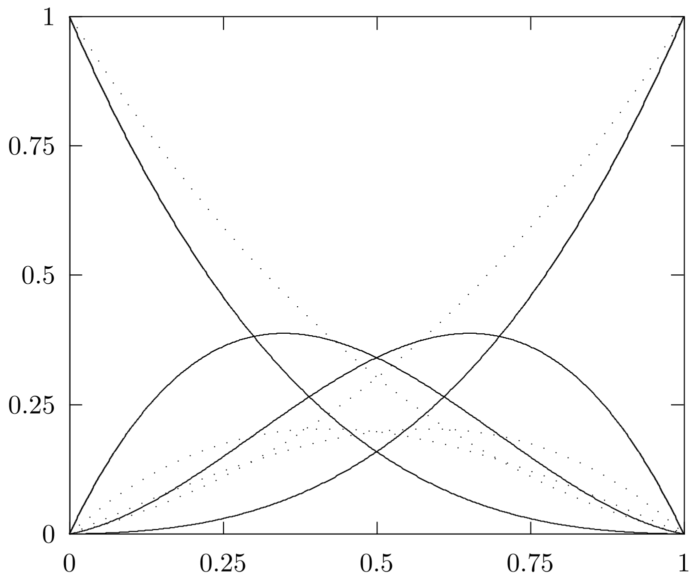

Fig. 3.1. The Stancu polynomials of degree 3 with different values for the shape parameter α (solid: $\alpha = 0.1$, dotted: $\alpha = 0.9$).

4. STANCU CURVES

The present section contains the major results. We introduce the Stancu curve scheme which generalize the Bézier curve scheme preserving all properties which are vital for applications in curve design. Moreover, the Stancu curve scheme provides more flexibility for the designer by means of the shape parameter α. We will see later (see Section 6), that the shape parameter can be implemented in a transparent manner into a CAD system controlling the Stancu scheme.

In the rest of the paper, α is considered a nonnegative parameter. We denote by $\mathbf{P}^n(\mathbb{R}, \mathbb{R}^m)$ the space of polynomials of degree at most n over $\mathbb{R}$ with values in $\mathbb{R}^m$.

Definition 4.1. *Let* $F \in \mathbf{P}^n(\mathbb{R}, \mathbb{R}^m)$ *an arbitrary polynomial. The representation*

$$F = \sum_{i=0}^{n} P_i S_i^n, \tag{4.1}$$

where $P_i \in \mathbb{R}^m$, $i = 0, \ldots, n$ *is called the* Stancu representation of the polynomial F on $[0,1]$. *If* F *admits such a representation, we say that the restriction of* F *to* $[0,1]$ *defines a* Stancu curve on $[0,1]$ with the shape parameter α. *The points* P_i, $i = 0, \ldots, n$ *are the* (Stancu) control points *and they define the* (Stancu) control polygon.

As with Bézier curves, we have the following evaluation algorithm for Stancu curves.

THEOREM 4.2. (De Casteljau Algorithm) *Consider the points* $P_j \in \mathbb{R}^m$, $j = 0, \ldots, n$, *where* $n \in \mathbb{N}$, $n \neq 0$. *Define the points* $P_{ij}^k \in \mathbb{R}^m$, $i + j + k = n$ *so that*

$$P_{ij}^0 := P_j, \quad P_{ij}^{k+1}(x) := \tfrac{v+i\alpha}{1+(i+j)\alpha} P_{i+1,j}^k(x) + \tfrac{u+j\alpha}{1+(i+j)\alpha} P_{i,j+1}^k \tag{4.2}$$

for $k = 0, \ldots, n-1$ *and* $x \in \mathbb{R}$. *Then* $P_{00}^n(x)$ *is a Stancu curve with the control points* P_j, $j = 0, \ldots, n$.

Proof. To show that P_{00}^n is a Stancu curve, we must derive the representation (4.1). We will show a bit more. We prove that the sums $s_k(x) = \sum_{i+j+k=n} P_{ij}^k(x) S_j^{n-k}(x)$ do not depend on $k = 0, \ldots, n$. Then we have

$$s_0(x) = \sum_{i+j=n} P_{ij}^0(x) S_j^n(x) = \sum_{j=0}^{n} P_j S_j^n(x), \text{ and}$$
$$s_n(x) = \sum_{i+j=0} P_{ij}^n(x) S_j^0(x) = P_{00}^n(x).$$

The equality $s_0(x) = s_n(x)$ gives the required representation.

To prove that the sums $s_k(x)$ are all the same for any $k = 0, \ldots, n$, we start with

$$\begin{aligned} s_k(x) &= \sum_{i+j+k=n} P_{ij}^k(x)\, S_j^{n-k}(x) \\ &= \sum_{i+j+k=n} P_{ij}^k(x) \left[\tfrac{v+(n-k-1-j)\alpha}{1+(n-k-1)\alpha}\, S_j^{n-k-1}(x) + \tfrac{u+(j-1)\alpha}{1+(n-k-1)\alpha}\, S_{j-1}^{n-k-1}(x)\right] = \end{aligned}$$

$$= \sum_{i+j+k=n} P_{ij}^k(x) \tfrac{v+(i-1)\alpha}{1+(i+j-1)\alpha}\, S_j^{n-k-1}(x) + \sum_{i+j+k=n} P_{ij}^k(x) \tfrac{u+(j-1)\alpha}{1+(i+j-1)\alpha}\, S_{j-1}^{n-k-1}(x).$$

Re-indexing the last sum we obtain

$$\begin{aligned} s_k(x) &= \sum_{i+j+k=n-1} P_{i+1,j}^k(x) \tfrac{v+i\alpha}{1+(i+j)\alpha}\, S_j^{n-k-1}(x) \\ &\quad + \sum_{i+j+k=n-1} P_{i,j+1}^k(x) \tfrac{u+j\alpha}{1+(i+j)\alpha}\, S_j^{n-k-1}(x) \\ &= \sum_{i+j+k=n-1} \left[\tfrac{v+i\alpha}{1+(i+j)\alpha} P_{i+1,j}^k(x) + \tfrac{u+j\alpha}{1+(i+j)\alpha} P^k(x)\right] S_j^{n-k-1}(x). \end{aligned}$$

Using (4.2) we have

$$s_k(x) = \sum_{i+j+k+1=n} P_{ij}^{k+1}(x) S_j^{n-(k+1)}(x) = s_{k+1}(x). \qquad \square$$

The de Casteljau algorithm is conveniently represented as a triangle array of computations, as illustrated in Figure 4.1. The functions along the edges of this graph are the coefficients on the right-hand side of the formula (4.2). Notice that if two arrows point into the same node, their labels sum to one. We can reverse the arrows in the triangle, place 1 at the apex of the triangle, run the computation downward and read the Stancu polynomials off the base of the triangle. This means that reversing the arrows, the triangle array represents the recursive computation formula (3.1) for the Stancu polynomials.

For $\alpha = 0$ both the recursive computation algorithm (3.1) for Stancu polynomials and the de Casteljau algorithm (4.2) for Stancu curves specialize to the Bernstein-Bézier case.

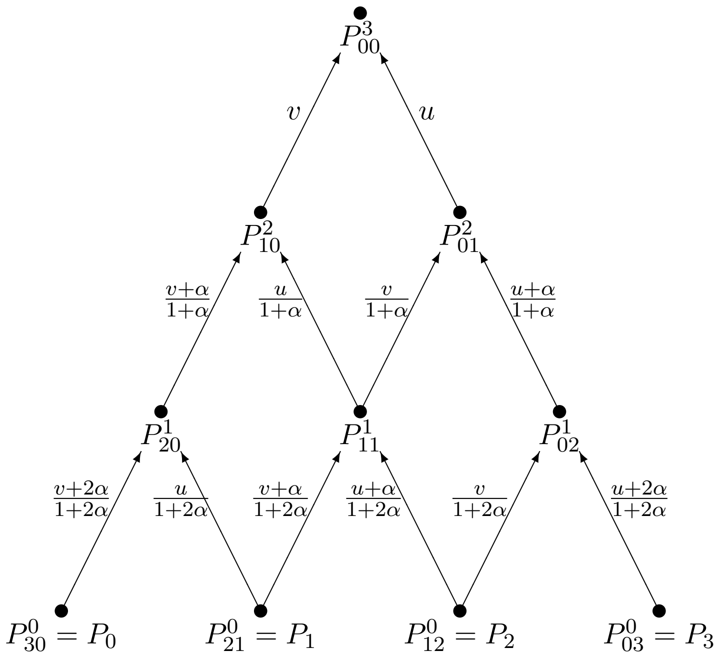

Fig. 4.1. The de Casteljau algorithm for a cubic Stancu curve.

Other important properties for Stancu curve are listed by the following corollary and follow easily from the properties of the Stancu polynomials.

Corollary 4.3. (1) *Let* I_{ij}^{k+1} *be the interval* $[-i\alpha, 1+j\alpha]$, $i+j+k = n$. *In the* k^{th} *step,* $k = 0, 1, \dots, n$, *of the de Casteljau algorithm the point* P_{ij}^{k+1} *divides the interval* $J_{ij}^{k+1} = [P_{i+1,j}^k(x), P_{i,j+1}^k(x)]$ *exactly in the same ratio as the point* x *divides the interval* I_{ij}^{k+1}.

(2) *Stancu curves are affine invariant.*
(3) *Stancu curves interpolate the end points of the control polygon, i.e.* $P^n_{00}(0) = P_0$ *and* $P^n_{00}(1) = P_n$.
(4) *Stancu curves are included in the convex hull of their control polygon.*

If the blending functions for a curve scheme form a basis of the whole space $\mathbf{P}^n(\mathbb{R}, \mathbb{R}^m)$ then any polynomial curve of degree at most n can be represented as a curve belonging to the scheme. We will show now that the Stancu polynomials form indeed a basis for the space of polynomials of degree at most n.

We recall from [3] the definition of Pólya polynomials, but see also [11], [1], and [2].

Definition 4.4. *Let* $t_1, \dots, t_n$ *be* $2n$ *real numbers such that* $t_{i+n} \neq t_j$ *for* $1 \leq i \leq j \leq n$ *(i.e.* $t_1, \dots, t_n$ *is a* progressive sequence*). The* n^{th} *degree polynomials*

$$p^n_i(x) = \prod_{j=i+1}^{n+i} (t_j - x), \quad i = 0, 1, \dots, n \tag{4.3}$$

are called the Pólya polynomials of degree n over the knot sequence $\bar{t} := \{t_1, \dots, t_n\}$.

The Stancu polynomials are special Pólya polynomials. More precisely, we have the following result.

Proposition 4.5. *Denote* $\bar{t} = \{t_j,\, j = 1, \dots, 2n\}$ *the knot sequence defined by*

$$t_j = \begin{cases} -(n-j)\alpha, & 1 \leq j \leq n \\ 1 + (j-n-1)\alpha, & n+1 \leq j \leq 2n \end{cases} \tag{4.4}$$

Up to coefficients, the Stancu polynomials are the Pólya polynomials over the knot sequence $\bar{t}$*, i.e.*

$$p^n_{n-i} = \frac{(-1)^i}{\left[{n \atop i}\right]} S^n_i,\ i = 0, 1, \dots, n. \tag{4.5}$$

Proof. We have successively

$$\begin{aligned} S^n_i(x) &= \left[{n \atop i}\right] u^{[i]} v^{[n-i]} = \left[{n \atop i}\right] \prod_{j=0}^{n-i-1} (1 + j\alpha - x) \prod_{j=0}^{i-1} (j\alpha + x) \\ &= (-1)^i \left[{n \atop i}\right] \prod_{k=n+1}^{2n-i} (1 + (k-n-1)\alpha - x) \prod_{j=n}^{n-i+1} (-(n-j)\alpha - x) \\ &= (-1)^i \left[{n \atop i}\right] \prod_{k=n+1}^{2n-i} (t_k - x) \prod_{k=n-i+1} n(t_k - x) \\ &= (-1)^i \left[{n \atop i}\right] \prod_{k=n-i+1}^{2n-i} (t_k - x) = (-1)^i \left[{n \atop i}\right] p^n_{n-i}(x). \end{aligned}$$

□

The Pólya polynomials p_{n-i}^n, $i=0,1,\dots,n$ form a basis for $\mathbf{P}^n(\mathbb{R},\mathbb{R}^m)$ [2], [3]. Hence the following theorem holds.

THEOREM 4.6. *Every polynomial* $F \in \mathbf{P}^n(\mathbb{R},\mathbb{R}^m)$ *admits a Stancu representation* $F=\sum_{i=0}^n P_i S_i^n$, *where* $P_i \in \mathbb{R}^m$, $i=0,1,\dots,n$ *are the control points of the Stancu curve defined by* F. *Starting from the control points* P_i, $i=0,1,\dots,n$ *the de Casteljau algorithm* (4.2) *evaluates the Stancu curve defined by* F *at any point* $x\in[0,1]$.

5. DEGREE ELEVATION OF STANCU CURVES

The Bézier curves admit a simple two-term degree elevation formula. Let $B=\sum_{i=0}^n P_i B_i^n$, $P_i\in\mathbb{R}^m$ be a Bézier curve of degree n, where B_i^n, $i=0,1,\dots,n$ denotes the Bernstein polynomials of degree n over $[0,1]$.

It is well known [6], [13] that the points $\bar{P}_i$, $i=0,1,\dots,n$ in the representation $B=\sum_{i=0}^{n+1}\bar{P}_i B_i^{n+1}$ of B as a Bézier curve of degree $n+1$ are given by

$$\bar{P}_i=\tfrac{i}{n+1}P_{i-1}+\big(1-\tfrac{i}{n+1}\big)P_i,\ i=0,1,\dots,n+1. \tag{5.1}$$

In other words, holding the end control points fixed : $\bar{P}_0=P_0$, $\bar{P}_{n+1}=P_{n+1}$ and substituting the $n-1$ inner control points P_i, $i=1,\dots,n-1$ by n points $\bar{P}_i$, $i=1,\dots,n$ with affine coordinates $(\frac{i}{n+1},1-\frac{i}{n+1})$, $i=1,\dots,n$, we obtain the control points of the same Bézier curve considered now as an $(n+1)^{\text{st}}$ degree curve. Note that the control points $\bar{P}_i$, $i=0,1,\dots,n+1$ depend only on the order they appear in the sequence $P_0,\dots,P_{n+1}$, i.e. only on the index i.

Despite of their greater flexibility provided by the shape parameter, the Stancu curves have, surprisingly, the very same two-term degree elevation formula as the Bézier curves. The explanation consists in the fact that the Stancu polynomials have similar degree elevation formulae to those of Bernstein polynomials. One easily see that

$$\tfrac{u+(n-i)\alpha}{1+n\alpha}\,S_i^n=\tfrac{i+1}{n+1}\,S_{i+1}^{n+1}, \tag{5.2}$$

$$\tfrac{v+i\alpha}{1+n\alpha}\,S_i^n=\tfrac{n+1-i}{n+1}\,S_i^{n+1} \tag{5.3}$$

and hence

$$S_i^n=\tfrac{n+1-i}{n+1}\,S_i^{n+1}+\tfrac{i+1}{n+1}\,S_{i+1}^{n+1} \tag{5.4}$$

for $i=0,1,\dots,n$. For $\alpha=0$ these formulae specialize to the well known degree elevation formulae for Bernstein polynomials:

$$u\,B_i^n=\tfrac{i+1}{n+1}\,B_{i+1}^{n+1}, \tag{5.5}$$

$$v\,B_i^n=\tfrac{n+1-i}{n+1}\,S_i^{n+1},\quad\text{and respectively} \tag{5.6}$$

$$B_i^n=\tfrac{n+1-i}{n+1}\,B_i^{n+1}+\tfrac{i+1}{n+1}\,B_{i+1}^{n+1} \tag{5.7}$$

for $i=0,1,\dots,n$.

To establish the degree elevation formula for Stancu curves, consider the n^{th} degree Stancu curve $S = \sum_{i=0}^{n} P_i S_i^n$. Expressing S as a $(n+1)^{\text{st}}$ degree curve we have $\sum_{i=0}^{n} P_i S_i^n = \sum_{i=0}^{n+1} \bar{P}_i S_i^{n+1}$. Observe that $\frac{v+i\alpha}{1+n\alpha} + \frac{u+(n-i)\alpha}{1+n\alpha} = 1$ and use (5.4) to get

$$\begin{aligned}\sum_{i=0}^{n} P_i S_i^n =& \sum_{i=0}^{n} P_i S_i^n \tfrac{v+i\alpha}{i+n\alpha} + \sum_{i=0}^{n} P_i S_i^n \tfrac{v+(n-i)\alpha}{1+n\alpha}\\ =& \sum_{i=0}^{n} P_i \tfrac{n+1-i}{n+1} S_i^{n+1} + \sum_{i=0}^{n} P_i \tfrac{i+1}{n+1} S_{i+1}^{n+1}\\ =& \sum_{i=0}^{n} P_i \tfrac{n+1-i}{n+1} S_i^{n+1} + \sum_{i=1}^{n+1} P_{i-1} \tfrac{i}{n+1} S_i^{n+1}\\ =& P_0 S_0^{n+1} + \sum_{i=1}^{n} \left[\tfrac{i}{n+1} P_{i-1} + (1 - \tfrac{i}{n+1}) P_i \right] S_i^{n+1} + P_{n+1} S_{n+1}^{n+1}.\end{aligned}$$

The last equality yields

Theorem 5.1. *Let $\sum_{i=0}^{n+1} \bar{P}_i S_i^{n+1}$ be the expression of the n^{th} degree Stancu curve $S = \sum_{i=0}^{n} P_i S_i^n$ as an $(n+1)^{\text{st}}$ degree Stancu curve. Then $\bar{P}_0 = P_0$, $\bar{P}_{n+1} = P_{n+1}$ and*

$$\bar{P}_i = \tfrac{i}{n+1} P_{i-1} + (1 - \tfrac{i}{n+1}) P_i, \quad i = 1, 2, \ldots, n. \tag{5.8}$$

6. CHANGE OF BASIS AND THE DERIVATIVE OF STANCU CURVES

The Bézier curves have a very simple derivative formula. Consider the Bézier curve $B = \sum_{i=0}^{n} P_i B_i^n$ with the control points $P_i \in \mathbb{R}^m$, $i = 0, 1, \ldots, n$. It is known [6], [13] that the derivative of the curve B is given by

$$\tfrac{\mathrm{d}}{\mathrm{d}x} B = n \sum_{i=0}^{n-1} (P_{i+1} - P_i)\, B_i^{n-1}. \tag{6.1}$$

From here follows that at the point $B(0) = P_0$ the curve is tangent to the vector $P_1 - P_0$, and at the point $B(1) = P_n$ the curve is tangent to the vector $P_n - P_{n-1}$.

Unfortunately there is no simple derivative formula for Stancu curves. The tangent vector at the first (last) control point depends not only on the first (last) two control points as in the case of Bézier curves, but rather on all control points. To establish the formula for the tangent vectors at the end points of a Stancu curve we will first transform the Stancu curve into a Bézier curve to obtain a relationship between the Stancu and the Bézier control points. This relationships leads us then to the formula for the tangent vectors.

But first we start with the following result about the Stancu polynomials.

Theorem 6.1. *Let* $S_i^n = \begin{bmatrix} n \\ i \end{bmatrix} u^{[i]} v^{[n-i]}$, $i = 0, 1, \ldots, n$ *be the* i^{th} *Stancu polynomial of degree* n *and let* α *be positive. Then for any* $x \in [0,1]$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}x} S_i^n(x) = \frac{n}{1+(n-1)\alpha} \left[\sum_{j=0}^{i-1} \frac{u+(i-1)\alpha}{i(u+j\alpha)} S_{i-1}^{n-1}(x) - \sum_{j=0}^{n-i-1} \frac{v+(n-i-1)\alpha}{(n-i)(v+j\alpha)} S_i^n(x) \right]. \tag{6.2}$$

Proof. Clearly,

$$\frac{\mathrm{d}}{\mathrm{d}x} S_i^n(x) = \begin{bmatrix} n \\ i \end{bmatrix} \left(v^{[n-i]} \frac{\mathrm{d}}{\mathrm{d}x} \left(u^{[i]} \right) + u^{[i]} \frac{\mathrm{d}}{\mathrm{d}x} \left(v^{[n-i]} \right) \right).$$

One can easily verify that

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(u^{[i]} \right) = \sum_{j=0}^{i-1} \frac{u+(i-1)\alpha}{u+j\alpha} u^{[i-1]}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(v^{[n-i]} \right) = - \sum_{j=0}^{i-1} \frac{v+(n-i-1)\alpha}{v+j\alpha} v^{[n-i-1]},$$

and hence

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} S_i^n(x) &= \begin{bmatrix} n \\ i \end{bmatrix} \left(\sum_{j=0}^{i-1} \frac{u+(i-1)\alpha}{u+j\alpha} u^{[i-1]} v^{[n-i]} - \sum_{j=0}^{i-1} \frac{v+(n-i-1)\alpha}{v+j\alpha} v^{[n-i-1]} u^{[i]} \right) \\ &= \frac{\begin{bmatrix} n \\ i \end{bmatrix}}{\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}} \sum_{j=0}^{i-1} \frac{u+(i-1)\alpha}{u+j\alpha} S_{i-1}^{n-1}(x) - \frac{\begin{bmatrix} n \\ i \end{bmatrix}}{\begin{bmatrix} n-1 \\ i \end{bmatrix}} \sum_{j=0}^{i-1} \frac{v+(n-i-1)\alpha}{v+j\alpha} S_i^{n-1}(x). \end{aligned}$$

Simplifying and re-ordering the right-hand side of the last equality we obtain (6.2):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} S_i^n(x) &= \frac{n}{i(1+(n-1)\alpha)} \sum_{j=0}^{i-1} \frac{u+(i-1)\alpha}{u+j\alpha} S_{i-1}^{n-1}(x) \\ &\quad - \frac{n}{(n-i)(1+(n-1)\alpha)} \sum_{j=0}^{n-i-1} \frac{v+(n-i-1)\alpha}{v+j\alpha} S_i^{n-1}(x) \\ &= \frac{n}{1+(n-1)\alpha} \left(\sum_{j=0}^{i-1} \frac{u+(i-1)\alpha}{i(u+j\alpha)} S_{i-1}^{n-1}(x) - \sum_{j=0}^{n-i-1} \frac{v+(n-i-1)\alpha}{(n-i)(v+j\alpha)} S_i^{n-1}(x) \right). \quad \square \end{aligned}$$

For $\alpha = 0$ the derivative formula for Stancu polynomials specializes to the derivative formula

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^n(x) = n \left[B_{i-1}^{n-1}(x) - B_i^{n-1}(x) \right]$$

for Bernstein polynomials.

We now turn our attention to the topic of transforming the Bernstein basis B_i^n, $i = 0, 1, \ldots, n$ into the Stancu basis S_i^n, $i = 0, 1, \ldots, n$. Our aim is to find a recurrence relation for the $(n+1)\times(n+1)$ matrix $M^n = \left[M^n_{ji}\right]$ such that

$$S^n_i = \sum_{j=0}^{n} M^n_{ji} B^n_j. \tag{6.3}$$

The following theorem re-establishes a known result [10] using only the definition of the matrix M and the recurrence formula (3.1) for Stancu curves.

THEOREM 6.2. *The transformation matrix M^n from the Bézier basis into the Stancu basis satisfies the following recurrence:*

$$\begin{aligned} M^{n+1}_{ji} = {} & \tfrac{j}{n+1}\left[\tfrac{1+(i-1)\alpha}{1+n\alpha} M^n_{j-i,i-1} + \tfrac{(n-i)\alpha}{1+n\alpha} M^n_{j-1,i}\right] \\ & + \tfrac{n-j+1}{n+1}\left[\tfrac{(i-1)\alpha}{1+n\alpha} M^n_{j,i-1} + \tfrac{1+(n-i)\alpha}{1+n\alpha} M^n_{j,i}\right], \end{aligned} \tag{6.4}$$

where the entries M^n_{kl} with $k,l \notin \{0,1,\ldots,n\}$ are considered null.

Proof. By definition

$$S^{n+1}_i = \sum_{j=0}^{n+1} M^{n+1}_{ji} B^{n+1}_j,\ i = 0,1,\ldots,n+1.$$

Applying the recurrence formula (3.1) for Stancu polynomials to the right-hand side and using the definition of M^n we get

$$\begin{aligned} S^{n+1}_i &= \tfrac{u+(i-1)\alpha}{1+n\alpha}\, S^n_{i-1} + \tfrac{v+(n-i)\alpha}{1+n\alpha}\, S^n_i \\ &= \tfrac{u+(i-1)\alpha}{1+n\alpha} \sum_{j=0}^{n} M^n_{j,i-1} B^n_j + \tfrac{v+(n-i)\alpha}{1+n\alpha} \sum_{j=0}^{n} M^n_{j,i} B^b_j \\ &= \sum_{j=0}^{n}\left[\tfrac{u+(i-1)\alpha}{1+n\alpha}\, M^n_{j,i-1} + \tfrac{v+(n-i)\alpha}{1+n\alpha}\, M^n_{j,i}\right] B^n_j. \end{aligned}$$

Using (5.5), (5.6) and (5.7) in the left-hand side of the last equality and regrouping we have

$$\begin{aligned} S^{n+1}_i = {} & \sum_{j=0}^{n} \tfrac{j+1}{n+1}\left[\tfrac{1+(i-1)\alpha}{1+n\alpha} M^n_{j,i-1} + \tfrac{(n-i\alpha}{1+n\alpha} M^n_{ji}\right] B^{n+1}_{j+1} \\ & + \sum_{j=0}^{n} \tfrac{n-j+1}{n+1}\left[\tfrac{(i-1)\alpha}{1+n\alpha} M^n_{j,i-1} + \tfrac{1+(n-i)\alpha}{1+n\alpha} M^n_{ji}\right] B^{n+1}_j. \end{aligned}$$

Now changing the summation index j to $j+1$ in the first sum, separating the term corresponding to $j = n+1$ in the first sum and the term corresponding to $j = 0$ in the second sum, and collecting all other term under the a common sum symbol, we get the recurrence relation. □

A very important property of the matrix M^n is given by the next theorem. We recall that a matrix $\left[A_{ij}\right]_{n\times n}$ is *stochastic* provided its elements are nonnegative and its rows form partitions of unity, i.e. $\sum_{j=0}^{n} A_{ij} = 1$, $i = 0,1,\ldots,n$.

Theorem 6.3. (see also Example 6 in [10]) *The transformation matrix M^n from the Bernstein basis into the Stancu basis is a stochastic matrix.*

Proof. For $n = 0$ the assertion is true. Suppose M^n is stochastic. Using the recurrence of the matrix M^n we get successively

$$\sum_{i=0}^{n+1} M_{ji}^{n+1} = \tfrac{j}{n+1} \sum_{i=0}^{n+1} \tfrac{1+(i-1)\alpha}{1+n\alpha} M_{j-1,i-1}^n + \tfrac{j}{n+1} \sum_{i=0}^{n+1} \tfrac{(n-1)\alpha}{2+n\alpha} M_{j-1,i}^n + \tfrac{n-j-1}{n+1} \sum_{i=0}^{n+1} \tfrac{(i-1)\alpha}{1+n\alpha} M_{j-1,i}^n + \tfrac{n-j-1}{n+1} \sum_{i=0}^{n+1} \tfrac{1+(n-i)\alpha}{1+n\alpha} M_{j,i}^n,$$

where the entries $M_{k,l}^n$ with $k, l \notin \{0, 1, \ldots, n\}$ are considered null. Hence

$$\sum_{i=0}^{n+1} M_{ji}^{n+1} = \tfrac{j}{n+1} \sum_{i=1}^{n+1} \tfrac{1+(i-1)\alpha}{1+n\alpha} M_{j-1,i-1}^n + \tfrac{j}{n+1} \sum_{i=0}^{n} \tfrac{(n-1)\alpha}{1+n\alpha} M_{j-1,i-1}^n + \tfrac{n-j-1}{n+1} \sum_{i=1}^{n+1} \tfrac{(i-1)\alpha}{1+n\alpha} M_{j,i-1}^n + \tfrac{n-j-1}{n+1} \sum_{i=1}^{n+1} \tfrac{(i-1)\alpha}{1+n\alpha} M_{j,i-1}^n + \tfrac{n-j-1}{n+1} \sum_{i=0}^{n} \tfrac{1+(n-i)\alpha}{1+n\alpha} M_{j,i}^n.$$

Re-indexing we get

$$\sum_{i=0}^{n+1} M_{ji}^{n+1} = \tfrac{j}{n+1} \sum_{i=0}^{n} M_{j-1,i}^n + \tfrac{n-j+1}{n+1} \sum_{i=0}^{n} M_{ji}^n = \tfrac{j}{n+1} + \tfrac{n-j+1}{n+1} = 1. \qquad \square$$

Now we are able to establish a derivative formula for Stancu curves.

Consider the Stancu curve $S = \sum_{i=0}^n P_i S_i^n$ and express the Stancu polynomials in the Bernstein basis using (6.3). Then

$$S = \sum_{i=0}^{n} P_i \sum_{j=0}^{n} M_{ji}^n B_j^n = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} M_{ji}^n P_i \right) B_j^n.$$

By the previous theorem the sum $\sum_{i=0}^n M_{ji}^n P_i$ is an affine combination of the Stancu control points P_i, $i = 0, 1, \ldots, n$, i.e. it defines a point $Q_j = \sum_{i=0}^n M_{ji}^n P_i, j = 0, 1, \ldots, n$ in $\mathbb{R}^m$. Clearly, $Q_j, j = 0, 1, \ldots, n$ are the Bézier control points for the Stancu curve S. We apply (6.1) to get a derivative formula:

$$\tfrac{\mathrm{d}}{\mathrm{d}x} S = n \sum_{j=0}^{n-1} (Q_{j+1} - Q_j)\, B_j^{n-1} = n \sum_{j=0}^{n-1} \left(\sum_{i=0}^{n} M_{j+1,i}^n P_i - \sum_{i=0}^{n} M_{ji}^n P_i \right) B_j^{n-1}.$$

This formula expresses the derivative of the Stancu curve S in terms of the Bernstein basis. One can convert the Bernstein polynomials back into the Stancu polynomials using the inverse of the matrix M^n to obtain the derivative of the curve S in terms of Stancu polynomials. We will not perform this transformation. Instead we turn to find an expression for the tangent vectors to the Stancu curve S at the end control points P_0 and P_n.

To this end, observe that $P_0 = Q_0$ and $P_n = Q_n$ by the interpolatory property of the Stancu and Bézier curves. For the transformation matrix M_n this means that the first row is the vector $M^n_{0,\bullet} = (1, 0, \ldots, 0)$ and the last row is the vector $M^n_{n,\bullet} = (0, \ldots, 0, 1)$. Calculating the derivative of the curve S for $x = 0$ and $x = 1$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x} S\Big|_{x=0} = n\left(\sum_{i=0}^{n} M^n_{1i} P_i - P_0\right)$$

and similarly

$$\frac{\mathrm{d}}{\mathrm{d}x} S\Big|_{x=1} = n\left(\sum_{i=0}^{n} M^n_{n-1,i} P_i - P_n\right)$$

respectively.

We collect this result into the next theorem.

Theorem 6.4. *Consider the Stancu curve $S = \sum^n P_i S^n_i$ and let M^n be the transformation matrix from the Bernstein basis $\{B^n_i,\ i = 0, 1, \ldots, n\}$ into the Stancu basis $\{S^n_i,\ i = 0, 1, \ldots, n\}$. At the point corresponding to the parameter value $x = 0$ the curve S is tangent to the vector $H_1 - P_0$, where H_1 is the point having the affine coordinates $(M^n_{10}, \ldots, M^n_{1n})$ relative to the control points. At the point corresponding to the parameter value $x = 1$ the curve S is tangent to the vector $H_{n-1} - P_n$, where H_{n-1} is the point having the affine coordinates $(M^n_{n-1,0}, \ldots, M^n_{n-1,n})$ relative to the control points.*

Note that for a given set of control points the points H_1 and H_{n-1} depend only on the shape parameter α. Hence, these points can be used as *handles* to model the curve. Acting on this points the designer is able to modify (implicitly and transparently) the value of the shape parameter α, modifying thus the shape of the curve.

We illustrate this for a Stancu curve of degree 3. For the transformation matrix the recurrence formula (6.4) gives

$$M^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M^2 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\alpha}{2(1+\alpha)} & \frac{1}{1+\alpha} & \frac{\alpha}{2(1+\alpha)} \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$M^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{3\alpha+4\alpha^2}{3(1+2\alpha)(1+\alpha)} & \frac{1}{1+2\alpha} & \frac{\alpha}{(1+2\alpha)(1+\alpha)} & \frac{2\alpha^2}{3(1+2\alpha)(1+\alpha)} \\ \frac{2\alpha^2}{3(1+2\alpha)(1+\alpha)} & \frac{\alpha}{(1+2\alpha)(1+\alpha)} & \frac{1}{1+2\alpha} & \frac{3\alpha+4\alpha^2}{3(1+2\alpha)(1+\alpha)} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the cubic Stancu curve with fixed control points P_0, P_1, P_2, and P_3 the handle points H_1 and H_2 are then given by

$$H_1 = \frac{3\alpha+4\alpha^2}{3(1+2\alpha)(1+\alpha)} P_0 + \frac{1}{1+2\alpha} P_1 + \frac{\alpha}{(1+2\alpha)(1+\alpha)} P_2 + \frac{2\alpha^2}{3(1+2\alpha)(1+\alpha)} P_3$$

and respectively

$$H_2 = \frac{2\alpha^2}{3(1+2\alpha)(1+\alpha)} P_0 + \frac{\alpha}{(1+2\alpha)(1+\alpha)} P_1 + \frac{1}{1+2\alpha} P_2 + \frac{3\alpha+4\alpha^2}{3(1+2\alpha)(1+\alpha)} P_3 \, .$$

Figure 6.1 shows a cubic Stancu curve for different values of the shape parameter α.

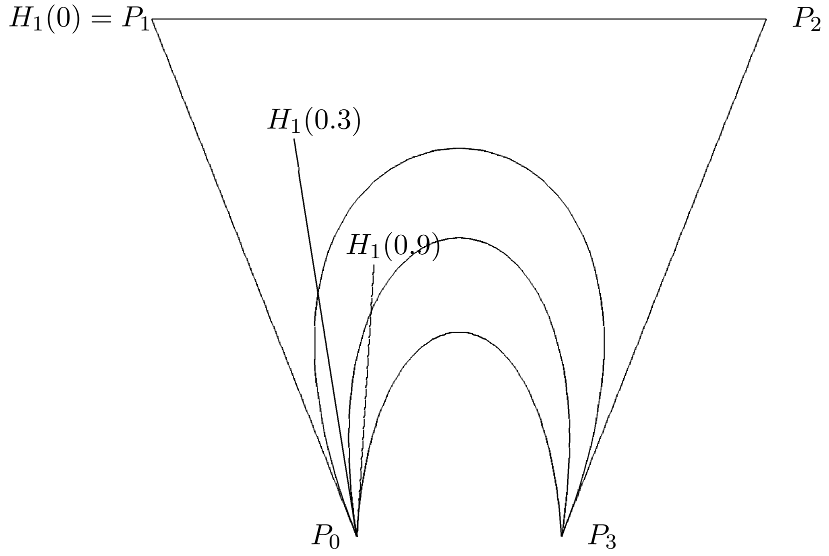

Fig. 6.1. A Stancu cubic for different values of the shape parameter α

7. CONCLUDING REMARKS AND FUTURE WORK

We have considered a class of polynomial curves based on the Stancu polynomial operator, the Stancu curves. These curves have a simple explicit formula and simple recurrence formula for the blending functions. Additionally, the Stancu curves have a shape parameter which may allow a designer to manipulate the shape of the curve without moving the control points, and to introduce such important effects as bias and tautness. These curves may be interesting for CAGD application. In some ways the Stancu curves are more flexible than the Bézier curves and the author used successfully these curves in ornamental design applications.

Because of the nice properties of the Stancu curves, it would be worthwhile to extend these results to surfaces. Such a generalization together with simplicial algorithms for Stancu curves and surfaces is currently under investigation [12].

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REFERENCES

[1] Barry, Ph. J., *Urn Models, Recursive curve schemes and Computer Aided Geometric Design*, Ph.D. thesis, Dept. of Math. Univ. of Utah, Salt Lake City, Utah, 1987.

[2] Barry, Ph. J. and Goldman, R. N., *Interpolation and approximation of curves and surfaces using Pólya polynomials*, Comput. Vision Graphics Image Process., **53**, no. 2, pp. 137–148, 1991.

[3] Barry, Ph. J., Goldman, R. N. and DeRose, T. D., *B-splines, Pólya curves and duality*, J. Approx. Theory, **65**, pp. 3–21, 1991.

[4] Eggenberger, F. and Pólya G., *Über die Statistik Verketteter Vorgänge*, Z. Angew. Math. Mech., **1**, pp. 279–289, 1923.

[5] Eisenberg, S. M. and Wood, B., *Approximation of analytic functions by Bernstein type operators*, J. Approx. Theory, **6**, pp. 242–248, 1972.

[6] Farin, G., *Curves and Surfaces for CAGD. A Practical Guide*, 3rd ed., Academic Press, 1993.

[7] Farin, G. and Barry, Ph. J., *Link between Bézier and Lagrange curve and surface schemes*, Computer-Aided Design, **18**, pp. 525–528, 1986.

[8] Mühlbach, G., *Verallgemeinerung der Bernstein- und Lagrange-Polynome. Bemerkungen zu einer Klasse linearer Polynomoperatoren von D. D. Stancu*, Rev. Roumaine Math. Pures Appl., **15**, pp. 1235–1252, 1970.

[9] Goldman, R. N., *Markov chains and Computer-Aided Geometric Design: Part I - Problems and Constraints*, ACM Trans. Graph., **3**, no. 3, pp. 204–222, 1984.

[10] Goldman, R. N., *Markov chains and Computer-Aided Geometric Design: Part II - Examples and subdivision matrices*, ACM Trans. Graph., **4**, no. 1, pp. 12–40, 1985.

[11] Goldman, R. N., *Pólya's Urn Model and Computer-Aided Geometric Design*, SIAM J. Algebraic Discrete Methods, **6**, pp. 1–28, 1985.

[12] Hatvany, A. Cs., *A simplicial approach to Stancu curves and surfaces*, in preparation.

[13] Hoschek, J. and Lasser, D., *Grundlagen der geometrischen Datenverarbeitung*, 2nd ed., Teubner, 1992.

[14] Mastroiani, G. and Occorsio, M. R., *Sulle derivate dei polinomi di Stancu*, Rend. Accad. Sci. Fis. Mat. Napoli, **4**, no. 45, pp. 273–281, 1978.

[15] Mastroiani, G. and Occorsio, M. R., *Una generalizaione dall'operatore di Stancu*, Rend. Accad. Sci. Fis. Mat. Napoli, **4**, no. 45, pp. 495–511, 1978.

[16] Stancu, D. D., *Approximations of functions by a new class of linear polynomial operators*, Rev. Roumaine Math. Pures Appl., **8**, pp. 1173–1194, 1968.

[17] Stancu, D. D., *Use of probabilistic methods in the theory of uniform approximation of continuous functions*, Rev. Roumaine Math. Pures Appl., **14**, pp. 673–691, 1969.

[18] Stancu, D. D., *Approximation properties of a class of linear positive operators*, Studia Univ. Babeş-Bolyai, Cluj, Ser. Math. Mech., **14**, pp. 33–38, 1970.

[19] Walz, G., *On Generalized Bernstein polynomials in CAGD*, Tech. Report 86, Univ. Mannheim, 1988.

[20] Walz, G., *Spline-Funktionen im Komplexen*, B. I. - Wissenschaftsverlag, 1991.
[21] Walz, G., *Tigonometric Bézier and Stancu polynomials over intervals and triangles*, CAGD, **14**, pp. 393–397, 1997.

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