REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 31 (2002) no. 2, pp. 195–198
ictp.acad.ro/jnaat

AITKEN-STEFFENSEN-TYPE METHODS FOR NONSMOOTH FUNCTIONS (II)*

ION PĂVĂLOIU†

Abstract. We present some new conditions which assure that the Aitken-Steffensen method yields bilateral approximation for the solution of a nonlinear scalar equation. The auxiliary functions appearing in the method are constructed under the hypothesis that the nonlinear application is not differentiable on an interval containing the solution.

MSC 2000. 65H05.
Keywords. Aitken-Steffensen iterations.

1. INTRODUCTION

In this note we continue the study of the convergence of the Aitken-Steffensen-type iterations

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(x_n); f]}, \quad n = 0, 1, \ldots, \quad x_0 \in I \tag{1}$$

for solving

$$f(x) = 0, \tag{2}$$

where $f : [a, b] \to \mathbb{R},\ a, b \in \mathbb{R},\ a < b$.

The functions g_1 and g_2 in (2) are chosen such that equations

$$x - g_i(x) = 0, \quad i = 1, 2, \tag{3}$$

to be equivalent to equation (1).

Under supplementary assumptions we shall show, as in [7], that (2) generates three monotone sequences, converging to the solution $\bar{x}$ of (1).

Regarding the monotonicity and convexity of f we shall consider the notions introduced in [3]. We shall also use Theorem 1 and Lemma 2 from [8].

For defining the functions g_1 and g_2, we shall consider $\alpha, \beta \in \mathbb{R}$ such that $a < \alpha < \beta < b$ and $f(\alpha) < 0,\ f(\beta) > 0$, defining then:

$$g_1(x) = x - \tfrac{f(x)}{[\beta, b; f]}, \quad x \in [\alpha, \beta], \tag{4}$$

$$g_2(x) = x - \tfrac{f(x)}{[a, \alpha; f]}, \quad x \in [\alpha, \beta]. \tag{5}$$

*This work has been supported by the Romanian Academy under grant GAR 45/2002.
†"T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68–1, 3400 Cluj-Napoca, Romania (pavaloiu@ictp.acad.ro).

Regarding f we shall make the following assumptions.

i. $f(\alpha) \cdot f(\beta) < 0$;
ii. f is increasing on $[a, b]$;
iii. f is convex on $[a, b]$ and continuous at a and b;
iv. if $\bar{x} \in (a, b)$ is the solution of (1), then f is differentiable at $\bar{x}$.

Remarks. 1° Hypothesiis iii. ensures the continuity of f on $[a, b]$, and therefore the existence of $\bar{x}$. Hypothesis ii. ensures the uniqueness of $\bar{x}$.

2° From hypotheses ii, iii and [8, Lm. 1.1], it follows that for any $u, v \in (\alpha, \beta)$ one obtains

$$[u, v; g_1] > 0 \quad \text{and} \quad [u, v; g_2] < 0, \tag{6}$$

i.e., g_1 is increasing and g_2 is decreasing on (α, β). □

Let $x_0 \in (\alpha, \beta)$ be such that

a) $f(x_0) < 0$
b) $g_2(x_0) < \beta$.

2. THE CONVERGENCE OF THE AITKEN-STEFFENSEN-TYPE ITERATIONS

We shall study in the following the convergence of the sequences $(x_n)_{n\geq 0}$ $(g_1(x_n))_{n\geq 0}$ to the solution $\bar{x}$. We obtain the following result.

Theorem 1. *If the function f verifies assumptions* i–iv., *the functions g_1 and g_2 are given by* (4) *and* (5), *and $x_0 \in (\alpha, \beta)$ verifies* a) *and* b), *then the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$ and $(g_2(x_n))_{n\geq 0}$, generated by* (2) *satisfy:*

j. $(x_n)_{n\geq 0}$ *is increasing;*
jj. $(g_1(x_n))_{n\geq 0}$ *is increasing;*
jjj. $(g_2(x_n))_{n\geq 0}$ *is decreasing;*
jv. *for all $n \in \mathbb{N}$, one has*

$$x_n < g_1(x_n) < x_{n+1} < \bar{x} < g_2(x_n). \tag{7}$$

Proof. By $[\beta, b; f] > 0$ and $f(x_0) < 0$ it follows $g_1(x_0) > x_0$, while by $x_0 < \bar{x}$ and the fact that g_1 is increasing it follows $g_1(x_0) < g_1(\bar{x}) = \bar{x}$, i.e. $g_1(x_0) < \bar{x}$.

Now, since $x_0 < \bar{x}$, g_2 is decreasing one gets $g_2(x_0) > g_2(\bar{x}) = \bar{x}$, i.e. the following relations hold:

$$x_0 < g_1(x_0) < \bar{x} < g_2(x_0). \tag{8}$$

Let $x_1 = g_1(x_0) - \frac{f(g_1(x_0))}{[g_1(x_0), g_2(x_0); f]}$, since $f(g_1(x_0)) < 0$ implies $g_1(x_0) < x_1$. From the identity

$$f(x_1) = f(g_1(x_0)) + [g_1(x_0), g_2(x_0); f](x_1 - g_1(x_0)) + \\ + [x_1, g_1(x_0), g_2(x_0); f](x_1 - g_1(x_0))(x_1 - g_2(x_0))$$

taking into account the convexity of f and relation (1), we get $f(x_1) < 0$, i.e. $x_1 < \bar{x}$.

By the above relations and by (8) it follows

$$x_0 < g_1(x_0) < x_1 < \bar{x} < g_2(x_0),$$

which shows that (7) is verified for $n = 0$.

Repeating this reason, the induction shows that (7) holds for $n \in \mathbb{N}$. This attracts in turn that the sequences $(x_n)_{n\geq 0}$ and $(g_1(x_n))_{n\geq 0}$ are increasing, i.e., statements j and jj.

We show next that $(g_2(x_n))_{n\geq 0}$ is decreasing. Indeed, by $x_n < x_{n+1}$ for $n \in \mathbb{N}$ we get $g_2(x_n) > g_2(x_{n+1})$ since g_2 is decreasing. Inequalities $x_n < \bar{x}$, $n \in \mathbb{N}$, show that $g_2(x_n) > g_2(\bar{x}) = \bar{x}$.

Let us notice that the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$, $(g_2(x_n))_{n\geq 0}$ are monotone and bounded, so they converge. Let $l_1 = \lim x_n$, $l_2 = \lim g_1(x_n)$ and $l_3 = \lim g_2(x_n)$. We show that $l_1 = l_2 = l_3 = \bar{x}$.

We prove first that $l_1 = l_2$. Assume the contrary, $l_1 \neq l_2$, e.g. $l_1 < l_2$. Obviously, $l_1 = \sup_{n\in\mathbb{N}} \{x_n\}$ and $l_2 = \sup_{n\in\mathbb{N}} \{g_1(x_n)\}$. Let $0 < \varepsilon < l_2 - l_1$ be a positive number. Then there exists $n_\varepsilon \in \mathbb{N}$ such that $g_1(x_n) > l_2 - \varepsilon$ for $n > n_\varepsilon$.

This implies that

$$x_{n+1} > g_1(x_n) > l_2 - \varepsilon > l_1$$

so l_1 is not the exact upper bound of the elements of the sequence $(x_n)_{n\geq 0}$. Hence, clearly, $l_1 = l_2 = l$,

$$l = \lim x_n = \lim g_1(x_n) = g_1(l),$$

i.e., $l = \bar{x}$. Since $\lim x_n = \bar{x}$, it follows that

$$\lim g_2(x_n) = g_2(\bar{x}) = \bar{x},$$

since $\bar{x}$ is the unique solution of equation $x - g_2(x) = 0$. □

The above relations show that we have a control of the error at each iteration step, justified by

$$\bar{x} - x_{n+1} < g_2(x_n) - x_{n+1}, \quad \text{or} \quad \bar{x} - x_{n+1} < g_2(x_n) - g_1(x_n), \quad n = 0, 1, \ldots$$

The identity

$$0 = f(\bar{x}) = f(g_1(x_n)) + [g_1(x_n), g_2(x_n); f](\bar{x} - g_1(x_n)) + \\ + [\bar{x}, g_1(x_n), g_2(x_n); f](\bar{x} - g_1(x_n))(\bar{x} - g_2(x_n))$$

relation (7), and the hypotheses of the above theorem lead to

$$\text{(9)} \qquad \bar{x} - x_{n+1} = -\frac{[\bar{x}, g_1(x_n), g_2(x_n); f]}{[g_1(x_n), g_2(x_n); f]}(\bar{x} - g_1(x_n))(\bar{x} - g_2(x_n)).$$

Further, by Lemma 2 from [8] we get

$$\bar{x} - g_1(x_n) = [x_n, \bar{x}; g_1](\bar{x} - x_n) \tag{10}$$

$$\bar{x} - g_2(x_n) = [x_n, \bar{x}, g_2](\bar{x} - x_n) \tag{11}$$

and, taking into account (4) and (5),

$$[x_n, \bar{x}; g_1] = 1 - \frac{[x_n,\bar{x};f]}{[\beta,b;f]} < 1 - \frac{[a,\alpha;f]}{[\beta,b;f]} = 1 - q < 1.$$

Analogously,

$$[\bar{x}, x_n; g_2] = 1 - \frac{[\bar{x},x_n;f]}{[a,\alpha;f]} > 1 - \frac{[\beta,b;f]}{[a,\alpha,;]} = \frac{[\beta,b;f]}{[a,\beta;f]}\left(\frac{[a,\alpha;f]}{[\beta,b;f]} - 1\right) = \frac{1}{q}(q-1)$$

where we have denoted $[a,\alpha;f] / [\beta,b;f] = q > 0$ and by Lemma 2 from [8] $q < 1$. This relation, together with the decreasing of g_2 lead to $-[\bar{x}, x_n; g_2] < \frac{1}{q}(1-q)$, i.e., $|[\bar{x}, x_n; g_2]| < \frac{1}{q}(1-q)$.

Denoting $M = \max_{u,v\in[\alpha.\beta]} |[\bar{x}, u, v; f]|$ and $m = [a, \alpha; f]$, by (9) we get

$$|\bar{x} - x_{n+1}| < \frac{M(1-q)^2}{mq} |\bar{x} - x_n|^2, \ n = 1, 2, \ldots$$

which characterizes the convergence order of the studied methods.

REFERENCES

[1] Balázs, M., *A bilateral approximating method for finding the real roots of real equations*, Rev. Anal. Numér. Théor. Approx., **21**, no. 2, pp. 111–117, 1992.

[2] Casulli, V. and Trigiante, D., *The convergence order for iterative multipoint procedures*, Calcolo, **13**, no. 1, pp. 25–44, 1977.

[3] Cobzaş, S., *Mathematical Analysis*, Presa Universitară Clujeană, Cluj-Napoca, 1997 (in Romanian).

[4] Ostrowski, A. M., *Solution of Equations and Systems of Equations*, Academic Press, New York, 1960.

[5] Păvăloiu, I., *On the monotonicity of the sequences of approximations obtained by Steffensens's method*, Mathematica (Cluj), **35** (**58**), no. 1, pp. 71–76, 1993.

[6] Păvăloiu, I., *Bilateral approximations for the solutions of scalar equations*, Rev. Anal. Numér. Théor. Approx., **23**, no. 1, pp. 95–100, 1994.

[7] Păvăloiu, I., *Approximation of the roots of equations by Aitken-Steffensen-type monotonic sequences*, Calcolo, **32**, no. 1–2, pp. 69–82, 1995.

[8] Păvăloiu, I., *Aitken-Steffensen-type methods for nonsmooth functions (I)*, Rev. Anal. Numér. Théor. Approx., **31**, no. 1, pp. 111–116, 2002.

[9] Traub, F. J., *Iterative Methods for the Solution of Equations*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

Received by the editors: January 16, 2002.