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GENERALIZED QUASICONVEX SET-VALUED MAPS

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Abstract. The aim of this paper is to introduce a concept of quasiconvexity for set-valued maps in a general framework, by only considering an abstract convexity structure in the domain and an arbitrary binary relation in the codomain. It is shown that this concept can be characterized in terms of usual quasiconvexity of certain real-valued functions. In particular, we focus on cone-quasiconvex set-valued maps with values in a partially ordered vector space.

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1. INTRODUCTION AND PRELIMINARIES

Several generalizations of the classical notions of convexity and quasiconvexity of real-valued functions have been given for vector-valued functions and even for set-valued maps with values in a partially ordered vector space (see e.g. [2]–[7]), the most natural of them being those which preserve the characteristic properties of convex and quasiconvex functions to have a convex epigraph and convex lower level sets, respectively. The aim of this paper is to extend the concept of quasiconvexity for set-valued maps in a very general framework: on one hand, the domain will be a set endowed by a convexity structure induced by a set-valued map Γ, the values of which will replace the linear segments; on the other hand, the order induced by a convex cone in the codomain will be replaced by a general binary relation Ω. The class of so-called (Γ, Ω)-quasiconvex set-valued maps will be introduced in Section 2. Then, by using a technique from [8], in Section 3 we shall characterize this class in terms of quasiconvexity of certain real-valued functions.

Let us firstly recall some notions of Set-Valued Analysis (see e.g. [1]). Given a set-valued map (i.e. a point to set function) $\Phi : A \rightsquigarrow B$ between some sets A and B, we denote by

$$\mathrm{Dom}(\Phi) = \{x \in A \ : \ \Phi(x) \neq \emptyset\}, \quad \mathrm{Graph}(\Phi) = \{(x, y) \in A \times B \ : \ y \in \Phi(x)\}$$

the domain and the graph of Φ, respectively.

A set-valued map $\Phi' : A \rightsquigarrow B$ is said to be an extension of Φ if $\mathrm{Graph}(\Phi) \subset \mathrm{Graph}(\Phi')$, which means that $\Phi(x) \subset \Phi'(x)$ for all $x \in A$.

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The inverse $\Phi^{-1} : B \rightsquigarrow A$ of Φ is the set-valued map defined for all $y \in B$ by

$$\Phi^{-1}(y) = \{x \in A \ : \ y \in \Phi(x)\}.$$

For any $U \subset A$ and $V \subset B$ the image of U by Φ and the inverse image of V by Φ are:

$$\Phi(U) = \bigcup\nolimits_{x \in U} \Phi(x) \quad \text{and} \quad \Phi^{-1}(V) = \{x \in A \ : \ \Phi(x) \cap V \neq \emptyset\}.$$

If $\Psi : B \rightsquigarrow C$ is a set-valued map, the composition product $\Psi \circ \Phi : A \rightsquigarrow C$ and the square product $\Psi \square \Phi : A \rightsquigarrow C$ of Ψ and Φ are the set-valued maps defined for all $x \in A$ by

$$\Psi \circ \Phi(x) = \bigcup\nolimits_{y \in \Phi(x)} \Psi(y) \quad \text{and} \quad \Psi \square \Phi(x) = \bigcap\nolimits_{y \in \Phi(x)} \Psi(y).$$

2. (Γ, Ω)-QUASICONVEX SET-VALUED MAPS

Throughout this paper X and Y will be two nonempty sets. We will introduce a class of generalized quasiconvex set-valued maps defined on X with values in Y.

To this aim, we endow the set X with an abstract convexity structure by means of a set-valued map $\Gamma : X \times X \rightsquigarrow X$, which assigns to each pair $(x_1, x_2) \in X \times X$ a subset $\Gamma(x_1, x_2)$ of X (i.e. a generalized segment). We say that a subset D of X is Γ-convex, if $\Gamma(D \times D) \subset D$. We also consider a set-valued map $\Delta : X \times X \rightsquigarrow X$ which assigns to each pair $(x_1, x_2) \in X \times X$ the set $\Delta(x_1, x_2) = \{x_1, x_2\}$.

On the other hand, we endow the set Y with a binary relation $\Omega \subset Y \times Y$, which will be regarded as a set-valued map $\Omega : Y \rightsquigarrow Y$, by identifying it with its graph.

DEFINITION 1. *A set-valued map $F : X \rightsquigarrow Y$ is called (Γ, Ω)-quasiconvex if the set-valued map $(\Omega^{-1} \circ F) \square \Gamma$ is an extension of $(\Omega^{-1} \circ F) \square \Delta$, i.e.*

$$(1) \quad \Omega^{-1}(F(x_1)) \cap \Omega^{-1}(F(x_2)) \subset \Omega^{-1}(F(x)), \quad \forall x_1, x_2 \in X, \ x \in \Gamma(x_1, x_2).$$

REMARK 1. If the binary relation Ω satisfies the following additional condition:

$$(2) \qquad \Omega^{-1}(y_1) \cap \Omega^{-1}(y_2) \neq \emptyset \quad \text{for all} \quad y_1, y_2 \in F(X),$$

then the domain Dom(F) is Γ-convex whenever condition (1) holds. $\square$

DEFINITION 2. *A function $f : D \to Y$, defined on a nonempty subset D of X, is called (Γ, Ω)-quasiconvex if the set-valued map $F : X \rightsquigarrow Y$, defined for all $x \in X$ by*

$$F(x) = \begin{cases} \{f(x)\} & \text{if} \quad x \in D \\ \emptyset & \text{if} \quad x \in X \setminus D, \end{cases}$$

is (Γ, Ω)-quasiconvex.

REMARK 2. If D is a nonempty Γ-convex subset of X, then a function $f : D \to Y$ is (Γ,Ω)-quasiconvex if and only if

$$\Omega^{-1}(f(x_1))\cap\Omega^{-1}(f(x_2)) \subset \Omega^{-1}(f(x)), \quad \forall x_1,x_2 \in D,\ x \in \Gamma(x_1,x_2). \tag{3}$$ □

EXAMPLE 1. Suppose that Y is a partially ordered vector space, with the order Ω induced by a convex cone $K \subset Y$, i.e. $K + K \subset \mathbb{R}_+K \subset K \neq \emptyset$ and

$$\Omega(y) = y - K \quad \text{for every} \quad y \in Y. \tag{4}$$

Then a set-valued map $F : X \rightsquigarrow Y$ is (Γ,Ω)-quasiconvex if and only if

$$(F(x_1) + K) \cap (F(x_2) + K) \subset F(x) + K, \quad \forall x_1,x_2 \in X,\ x \in \Gamma(x_1,x_2).$$

Note that if the cone K generates Y, i.e. $K - K = Y$, then condition (2) is fulfilled.

In particular, if X is a vector space and Γ is the usual convex hull, i.e.

$$\Gamma(x_1,x_2) = \{tx_1 + (1-t)x_2 \ :\ t \in [0,1]\} \quad \text{for all} \quad x_1,x_2 \in X, \tag{5}$$

then $F : X \rightsquigarrow Y$ is (Γ,Ω)-quasiconvex if and only if

$$(F(x_1) + K) \cap (F(x_2) + K) \subset F(tx_1 + (1-t)x_2) + K, \tag{6}$$

for all $x_1,x_2 \in X,\ t \in [0,1]$.

According to Theorem 3 below, condition (6) actually means that F is K-quasiconvex in the sense of Kuroiwa [5]. Moreover, if $f : D \to Y$ is a vector-valued function defined on some nonempty convex subset D of X, then f is (Γ,Ω)-quasiconvex in the sense of Definition 2 if and only if it is K-quasiconvex in the sense of Dinh The Luc [6]. Finally, for $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, we recover the classical notion of quasiconvexity of real-valued functions. □

The following result shows that (Γ,Ω)-quasiconvexity naturally extends the classical notion of quasiconvexity, since it can be characterized in terms of certain generalized convex level sets (see e.g. [9] for other generalizations based on convex level sets).

THEOREM 3. *A set-valued map $F : X \rightsquigarrow Y$ is (Γ,Ω)-quasiconvex if and only if for every $y \in Y$ the generalized level set $F^{-1}(\Omega(y)) = \{x \in X \ :\ F(x) \cap \Omega(y) \neq \emptyset\}$ is Γ-convex.*

Proof. Firstly observe that $F^{-1}(\Omega(y)) = \{x \in X \ :\ y \in \Omega^{-1}(F(x))\}$ for all $y \in Y$.

Now, assume that $F : X \rightsquigarrow Y$ is (Γ,Ω)-quasiconvex and fix an arbitrary $y \in Y$. Let $x_1,x_2 \in F^{-1}(\Omega(y))$. Then we have $y \in \Omega^{-1}(F(x_1)) \cap \Omega^{-1}(F(x_2))$ and by (1) it follows that $y \in \Omega^{-1}(F(x))$ for all $x \in \Gamma(x_1,x_2)$, which means that $\Gamma(x_1,x_2) \subset F^{-1}(\Omega(y))$. Thus the set $F^{-1}(\Omega(y))$ is Γ-convex.

Conversely, suppose that for each $y \in Y$ the set $F^{-1}(\Omega(y))$ is Γ-convex. Let $x_1,x_2 \in X$ and $y_0 \in \Omega^{-1}(F(x_1)) \cap \Omega^{-1}(F(x_2))$. Then $x_1 \in F^{-1}(\Omega(y_0))$ and $x_2 \in F^{-1}(\Omega(y_0))$. Since the set $F^{-1}(\Omega(y_0))$ is Γ-convex, we infer that $\Gamma(x_1,x_2) \subset F^{-1}(\Omega(y_0))$, i.e. $y_0 \in \Omega^{-1}(F(x))$ for all $x \in \Gamma(x_1,x_2)$. Thus (1) holds. □

Corollary 4. *A function $f : D \to Y$, defined on a nonempty Γ-convex subset D of X, is (Γ, Ω)-quasiconvex if and only if the level set $f^{-1}(\Omega(y)) = \{x \in D \ : \ f(x) \in \Omega(y)\}$ is Γ-convex, for every $y \in Y$.*

Proof. In view of Remark 2, the conclusion directly follows from Theorem 3. $\square$

To conclude this section, consider the particular case where $Y = \mathbb{R}$ is endowed with the usual order relation Ω_u, defined by (4) with $K = \mathbb{R}_+$, i.e.

$$\Omega_u(y) = y - \mathbb{R}_+ = \{z \in \mathbb{R} \ : \ z \leq y\} \quad \text{for all} \quad y \in \mathbb{R}.$$

For any set-valued map $G : X \rightsquigarrow \mathbb{R}$ with nonempty compact values, we denote by $\mu_G : X \to \mathbb{R}$ the lower marginal function of G, defined for all $x \in X$ by

$$\mu_G(x) = \inf\, G(x).$$

Lemma 5. *Let $G : X \rightsquigarrow \mathbb{R}$ be a set-valued map with nonempty compact values. Then G is (Γ, Ω_u)-quasiconvex if and only if the lower marginal function $\mu_G : X \to \mathbb{R}$ is (Γ, Ω_u)-quasiconvex.*

Proof. By Theorem 3, G is (Γ, Ω_u)-quasiconvex if and only if for every $y \in \mathbb{R}$ the set $G^{-1}(\Omega_u(y))$ is Γ-convex. On the other hand, by Corollary 4 (note that the domain X of μ_G is Γ-convex) it follows that μ_G is (Γ, Ω_u)-quasiconvex if and only if for each $y \in \mathbb{R}$ the set $\mu_G^{-1}(\Omega_u(y))$ is Γ-convex. Actually, for all $y \in \mathbb{R}$ we have:

$$\begin{aligned} G^{-1}(\Omega_u(y)) &= \{x \in X \ : \ G(x) \cap \Omega_u(y) \neq \emptyset\} \\ &= \{x \in X \ : \ \exists\, z \in G(x) \ \text{s.t.} \ z \leq y\} \\ &= \{x \in X \ : \ \inf\, G(x) \leq y\} \\ &= \{x \in X \ : \ \mu_G(x) \in \Omega_u(y)\} \ = \ \mu_G^{-1}(\Omega_u(y)), \end{aligned}$$

since $G(x)$ is nonempty compact for every $x \in X$. Thus the desired equivalence holds. $\square$

3. (Γ, Ω)-QUASICONVEXITY VIA (Γ, Ω_U)-QUASICONVEXITY

The aim of this section is to characterize (Γ, Ω)-quasiconvex set-valued maps in terms of certain (Γ, Ω_u)-quasiconvex set-valued maps and their lower marginal functions. As in [8], our approach is essentially based on the concept of properly characteristic function associated to a binary relation.

Definition 6. *A function $\omega : Y \times Y \to \mathbb{R}$ is said to be:*

(i) *characteristic for Ω, if for any $y, z \in Y$ we have*

$$\omega(y,z) \leq 0 \quad \textit{if and only if} \quad y \in \Omega(z),$$

i.e., $\mathrm{Graph}(\Omega^{-1}) = \{(y,z) \in Y \times Y \ : \ \omega(y,z) \leq 0\}$.

(ii) *properly characteristic for* Ω, *if in addition to (i) there exists* $\hat{z} \in Y$ *such that*

$$\max\{\omega(y_1,\hat{z}),\, \omega(y_2,\hat{z})\} \leq 0 < \omega(y_3,\hat{z}),$$

whenever $\max\{\omega(y_1,z), \omega(y_2,z)\} < \omega(y_3,z)$, *for some* $y_1,y_2,y_3,z \in Y$.

Example 2. The function $\omega : Y \times Y \to \mathbb{R}$ defined for all $(y,z) \in Y \times Y$ by

$$\omega(y,z) = \begin{cases} 0, & \text{if} \quad y \in \Omega(z) \\ 1, & \text{if} \quad y \in Y \setminus \Omega(z) \end{cases}$$

is properly characteristic for Ω. □

Example 3. Let Y be a topological vector space, partially ordered by a closed convex cone K with nonempty interior, and let Ω be given by (4). As shown in [8], for any fixed point $e \in \mathrm{int}\, K$, the function $\omega : Y \times Y \to \mathbb{R}$ defined for all $(y,z) \in Y \times Y$ by

$$\omega(y,z) = \inf\{t \in \mathbb{R} \ : \ y \in \Omega(z + te) = z + te - K\} \tag{7}$$

is properly characteristic for Ω. In this case, for any fixed $z \in Y$, the function $\omega(\cdot,z) : Y \to \mathbb{R}$ represents the "smallest strictly monotonic function at z" in the sense of Dinh The Luc [6]. Note that this function is continuous (this property will be used further to obtain an application of Corollary 9). □

Given a function $\omega : Y \times Y \to \mathbb{R}$ and a set-valued map $F : X \rightsquigarrow Y$, for each $z \in Y$ we denote by $\omega(F(\cdot),z) : X \rightsquigarrow \mathbb{R}$ the set-valued map defined for all $x \in X$ by

$$\omega(F(x),z) = \{\omega(y,z) \ : \ y \in F(x)\}.$$

Theorem 7. *Let* $F : X \rightsquigarrow Y$ *be a set-valued map. If the function* $\omega : Y \times Y \to \mathbb{R}$ *is characteristic for* Ω *and the set-valued map* $\omega(F(\cdot),z)$ *is* (Γ,Ω_u)*-quasiconvex for each* $z \in Y$*, then* F *is* (Γ,Ω)*-quasiconvex.*

Proof. Suppose to the contrary that F is not (Γ,Ω)-quasiconvex. Then there exist some $x_1,x_2 \in X$, $z \in \Omega^{-1}(F(x_1)) \cap \Omega^{-1}(F(x_2))$ and $x_0 \in \Gamma(x_1,x_2)$ with $z \notin \Omega^{-1}(F(x_0))$. Hence

$$\Omega(z) \cap F(x_1) \neq \emptyset, \quad \Omega(z) \cap F(x_2) \neq \emptyset, \text{ and} \tag{8}$$

$$\Omega(z) \cap F(x_0) = \emptyset. \tag{9}$$

By (8) we infer the existence of $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ with $y_1 \in \Omega(z)$ and $y_2 \in \Omega(z)$. Since ω is characteristic for Ω, it follows that $\omega(y_1,z) \leq 0$ and $\omega(y_2,z) \leq 0$, which means that $0 \in \Omega_u^{-1}(\omega(y_1,z)) \cap \Omega_u^{-1}(\omega(y_2,z))$. Hence $0 \in \Omega_u^{-1}(\omega(F(x_1),z)) \cap \Omega_u^{-1}(\omega(F(x_2),z))$. Taking into account that $\omega(F(\cdot),z)$ is (Γ,Ω_u)-quasiconvex and recalling that $x_0 \in \Gamma(x_1,x_2)$ we can deduce that $0 \in \Omega_u^{-1}(\omega(F(x_0),z))$. This means that there exists some $y_0 \in F(x_0)$ such that $0 \in \Omega_u^{-1}(\omega(y_0, z))$, i.e. $\omega(y_0, z) \le 0$. The function ω being characteristic for Ω we infer that $y_0 \in \Omega(z) \cap F(x_0)$, i.e. a contradiction with (9). $\square$

Theorem 8. *Let $F : X \rightsquigarrow Y$ be a set-valued map and let $\omega : Y \times Y \to \mathbb{R}$ be a properly characteristic function for Ω. Assume that the following condition holds:*

$(\mathcal{C})$ *For every $x \in X$ there exists $y_x \in F(x)$ such that*

$$\omega(y_x, z) \le \omega(y, z) \quad \text{for all} \quad y \in F(x),\ z \in Y. \tag{10}$$

Then the following assertions are equivalent:

(i) *F is (Γ, Ω)-quasiconvex;*
(ii) *$\omega(F(\cdot), z)$ is (Γ, Ω_u)-quasiconvex for each $z \in Y$.*

Proof. The implication $(ii) \Rightarrow (i)$ was already proven in Theorem 7. In order to prove the converse implication, assume that F is (Γ, Ω)-quasiconvex and suppose to the contrary that $\omega(F(\cdot), \tilde{z})$ is not (Γ, Ω_u)-quasiconvex for a certain $\tilde{z} \in Y$. Then there exist $x_1, x_2 \in X$ and $x_3 \in \Gamma(x_1, x_2)$ such that $\Omega_u^{-1}(\omega(F(x_1), \tilde{z})) \cap \Omega_u^{-1}(\omega(F(x_2), \tilde{z})) \not\subset \Omega_u^{-1}(\omega(F(x_3), \tilde{z}))$. Thus we can choose a number $\alpha \in (\omega(F(x_1), \tilde{z}) + \mathbb{R}_+) \cap (\omega(F(x_2), \tilde{z}) + \mathbb{R}_+) \setminus (\omega(F(x_3), \tilde{z}) + \mathbb{R}_+)$. Hence there are some points $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ with

$$\max\{\omega(y_1, \tilde{z}), \omega(y_2, \tilde{z})\} \le \alpha < \omega(y, \tilde{z}) \quad \text{for all} \quad y \in F(x_3). \tag{11}$$

Let $y_{x_3} \in F(x_3)$ be a point which satisfies (10) with $x = x_3$. Then, by applying (11) for $y = y_{x_3}$, it follows that $\max\{\omega(y_1, \tilde{z}), \omega(y_2, \tilde{z})\} < \omega(y_{x_3}, \tilde{z})$ and, by taking into account that function ω is properly characteristic for Ω, we infer the existence of some $\hat{z} \in Y$ such that

$$\max\{\omega(y_1, \hat{z}), \omega(y_2, \hat{z})\} \le 0 < \omega(y_{x_3}, \hat{z}). \tag{12}$$

On the other hand, by using (10) with $x = x_3$ and $z = \hat{z}$, we obtain that $\omega(y_{x_3}, \hat{z}) \le \omega(y, \hat{z})$ for all $y \in F(x_3)$. Thus, by (12) we infer that

$$\max\{\omega(y_1, \hat{z}), \omega(y_2, \hat{z})\} \le 0 < \omega(y, \hat{z}) \quad \text{for all} \quad y \in F(x_3).$$

Since ω is characteristic for Ω, the above condition means that $y_1 \in \Omega(\hat{z})$, $y_2 \in \Omega(\hat{z})$ and $y \in Y \setminus \Omega(\hat{z})$ for all $y \in F(x_3)$. Hence

$$\hat{z} \in \Omega^{-1}(F(x_1)) \cap \Omega^{-1}(F(x_2)) \setminus \Omega^{-1}(F(x_3)),$$

which yields

$$\Omega^{-1}(F(x_1)) \cap \Omega^{-1}(F(x_2)) \not\subset \Omega^{-1}(F(x_3)),$$

i.e. a contradiction with the hypothesis (i). $\square$

Remark 3. If $F : X \rightsquigarrow Y$ is single-valued, i.e. if $F(x) = \{f(x)\}$ for all $x \in X$, where $f : X \to Y$ is a function, then condition $(\mathcal{C})$ becomes trivial. Therefore Theorem 8 extends similar results from [6]–[8], which have been obtained for single-valued functions. $\square$

Remark 4. If F is not single-valued, some additional assumptions must be imposed on F and ω in order to ensure $(\mathcal{C})$, as for example conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ below:

$(\mathcal{C}_1)$ For each $x \in X$, there exists a smallest element y_x in $F(x)$ with respect to Ω, i.e.

$$y_x \in F(x) \cap \Omega(y) \quad \text{for all} \quad y \in F(x);$$

$(\mathcal{C}_2)$ For each $z \in Y$, the function $\omega(\cdot, z) : Y \to \mathbb{R}$ is monotonic with respect to Ω, i.e.

$$\omega(y', z) \leq \omega(y, z) \quad \text{for all} \quad y \in Y,\ y' \in \Omega(y).$$

Note that in the particular case presented in Example 3, the function ω given by (7) satisfies condition $(\mathcal{C}_2)$. In this case, by imposing the assumption $(\mathcal{C}_1)$ on F we ensure $(\mathcal{C})$ in Theorem 8 (see Corollary 9 for an application). □

Remark 5. If the function ω is characteristic, but not properly characteristic for Ω, the implication $(i) \Rightarrow (ii)$ in Theorem 8 fails to be true, even if condition $(\mathcal{C})$ is fulfilled.

Indeed, consider the particular case where $X = [-1, 1]$ is endowed with the classical Γ given by (5) and let $Y = \mathbb{R}^2$ be endowed with the order relation Ω given by (4) with $K = \mathbb{R}^2_+$. As shown in [8], the function $\omega : Y \times Y \to \mathbb{R}$, defined by

$$\omega(y, z) = \begin{cases} -1, & \text{if } y \in \Omega(z) \setminus \Omega^{-1}(z) \\ 0, & \text{if } y \in \Omega(z) \cap \Omega^{-1}(z) \\ 1, & \text{if } y \in Y \setminus \Omega(z) \end{cases}$$

is characteristic, but not properly characteristic for Ω. Consider the vector-valued function $f = (f_1, f_2) : X \to Y$ defined for all $x \in X$ by

$$f(x) = (f_1(x), f_2(x)) = \begin{cases} (0, x), & \text{if } x \in [-1, 0] \\ (-x, 0), & \text{if } x \in [0, 1]. \end{cases}$$

It can be easily seen that f is K-quasiconvex, since its scalar components $f_1 : X \to \mathbb{R}$ and $f_2 : X \to \mathbb{R}$ are quasiconvex in the classical sense (see e.g. [6] and [3] for a detailed study of K-quasiconvex vector-valued functions). Hence the single-valued map $F : X \rightsquigarrow Y$, defined for all $x \in X$ by $F(x) = \{f(x)\}$, is (Γ, Ω)-quasiconvex. On the other hand, in view of Remark 3, condition $(\mathcal{C})$ holds. However, by choosing the point $z = (0, 0)$, it is a simple exercise to check that the set-valued map $\omega(F(\cdot), z) : X \rightsquigarrow \mathbb{R}$ is given by

$$\omega(F(x), z) = \{\varphi(x)\} = \begin{cases} \{-1\}, & \text{if } x \in [-1, 1] \setminus \{0\} \\ \{0\}, & \text{if } x = 0, \end{cases}$$

which is not (Γ, Ω_u)-quasiconvex, since the associate function $\varphi : X \to \mathbb{R}$ is not quasiconvex in the classical sense. □

We conclude by presenting a characterization of (Γ, Ω)-quasiconvex set-valued map with values in a topological space in terms of (Γ, Ω_u)-quasiconvexity of real-valued functions.

COROLLARY 9. *In addition to the hypotheses of Theorem* 8, *assume that: Y is a nonempty topological space, F has nonempty compact values, and ω is continuous with respect to the first argument. Then F is (Γ, Ω)-quasiconvex if and only if for each $z \in Y$ the lower marginal function of the set-valued map $\omega(F(\cdot), z)$ is (Γ, Ω_u)-quasiconvex.*

Proof. Since F has nonempty compact values and ω is continuous with respect to the first argument, it follows that for every $z \in Y$ the set-valued map $\omega(F(\cdot), z)$ has nonempty compact values. Hence the conclusion follows directly from Lemma 5 and Theorem 8. □

Note that, in view of Example 3 and Remark 4, Corollary 9 may be applied to characterize those (Γ, Ω)-quasiconvex set-valued maps with values in a topological ordered vector space, which have nonempty compact values, each value containing a smallest element.

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