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ON THE CONVERGENCE OF A METHOD FOR SOLVING TWO POINT BOUNDARY VALUE PROBLEMS BY OPTIMAL CONTROL

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Abstract. Using the idea of the least squares method, a nonlinear two point boundary value problem is transformed into an optimal control problem. For solving the optimal control problem it is used the gradient method. The convergence of the method is investigated and numerical results are reported.

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1. INTRODUCTION

In this paper we study the convergence property of a method to solve the nonlinear two point boundary value problem (NTPBVP)

$$x^{(m)}(t) = f(x(t), \dot{x}(t), \ldots, x^{(m-1)}(t), t), \qquad t \in [a, b], \tag{1}$$

$$\sum_{j=1}^{m} \left[\alpha_{i,j} x^{(j-1)}(a) + \beta_{i,j} x^{(j-1)}(b)\right] = \gamma_i, \qquad i \in \{1, 2, \ldots, m\} \tag{2}$$

using an optimal control problem (OCP). For the problem

$$\ddot{x}(t) = f(x(t), \dot{x}(t), t), \qquad t \in [0, T], \tag{3}$$

$$x(0) = \alpha, \tag{4}$$

$$x(T) = \beta, \tag{5}$$

the method was described in a previous note [11].

Sokolowski, Matsumura and Sakawa [12] used optimal control methods to solve two point boundary value problems of the form

$$-\tfrac{\mathrm{d}}{\mathrm{d}t}\Big[a\big(t, y(t), \tfrac{\mathrm{d}y}{\mathrm{d}t}\big)\tfrac{\mathrm{d}y}{\mathrm{d}t}\Big] + qy(t) = f(t), \quad t \in [0, 1],$$

$$y(0) = y(1) = 0.$$

The nonlinear two point boundary value problems and the optimal control problems are connected. The necessary optimality conditions, as Pontryagin's

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maximum principle, lead for some optimal control problem to a nonlinear two point boundary value problem such as (3)–(5).

The multiple shooting method (Keller H. B. [6], Marzulli P., [8]), the collocation method (Ascher U., Christiansen J., Russell R. D., [1], [2]) are well known and widely used to solve a NTPBVP.

In our case, the derived OCP may be solved efficiently using the gradient method. The application of the gradient method to solve optimal control problems is well known: Polak E. [10], Polak E., Klessig R., 1973; Fedorenko P. R., 1878 and Miele A. [9].

Another possible method to solve the optimal control problem is the control parametrization (Goh C. Z., Teo K. L., 1988, Teo K. L., Goh C. J., Wong K. H., 1991).

Although the NTPBVP (1)-(2) has not a very general form, thanks to the boundary conditions, our approach emphasizes a class of NTPBVP which may be efficiently solved using optimization techniques.

2. STATEMENT OF THE PROBLEM

Consider the NTPBVP (1)-(2).

We assume that the NTPBVP has an unique solution and that f is continuous together with his partial derivates of first and second order. If $x(t)$ is the solution of the NTPBVP (1)-(2) then the pair $(u(t), x(t))$ is the solution of the following OCP

$$\text{(6)} \qquad \textit{minimize} \quad I(u) = \int_a^b \left[u(t) - f(x(t), \dot{x}(t), \ldots, x^{(m-1)}(t), t)\right]^2 \mathrm{d}t$$

subject to

$$\text{(7)} \qquad x^{(m)}(t) = u(t), \qquad t \in [a, b],$$

and (2).

Denoting $x_1(t) = x(t),\ x_2(t) = \dot{x}(t),\ \ldots\ x_m(t) = \dot{x}_{m-1}(t),\ u(t) = \dot{x}_m(t)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, the above problem may be written as an OCP for a first order differential system:

$$\text{(8)} \qquad \textit{minimize} \quad I(u) = \int_a^b \left[u(t) - f(x_1(t), x_2(t), \ldots, x_m(t), t)\right]^2 \mathrm{d}t$$

subject to

$$\text{(9)} \qquad \dot{\mathbf{x}}(t) = Q\,\mathbf{x}(t) + \xi_m\, u(t), \qquad t \in [a, b],$$

$$\text{(10)} \qquad A\,\mathbf{x}(a) + B\,\mathbf{x}(b) = \gamma,$$

where

$$Q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\xi_m = (0, 0, \dots, 0, 1)^t,$$
$$A = (\alpha_{i,j})_{1 \le i,j \le m},$$
$$B = (\beta_{i,j})_{1 \le i,j \le m},$$
$$\gamma = (\gamma_i)_{1 \le i \le m}.$$

For given u the solution of the linear system (9) is

$$\mathbf{x}^u(t) = H(t)c + \int_a^t \varphi(t,s)u(s)\mathrm{d}s, \tag{11}$$

where

$$H(t) = \begin{pmatrix} 1 & \frac{t-a}{1!} & \frac{(t-a)^2}{2!} & \dots & \frac{(t-a)^{m-1}}{(m-1)!} \\ 0 & 1 & \frac{t-a}{1!} & \dots & \frac{(t-a)^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

$$\varphi(t,s) = \left(\tfrac{(t-s)^{m-1}}{(m-1)!},\ \tfrac{(t-s)^{m-2}}{(m-2)!},\ \dots,\ 1 \right)^t,$$
$$c = (c_0,\ c_1,\ \dots,\ c_{m-1})^t.$$

Using the shooting method, in order to satisfy the boundary condition (10), the vector c is the solution of the following algebraical system

$$[A + BH(b)]c = \gamma - B\int_a^b \varphi(b,s)u(s)\mathrm{d}s.$$

We suppose that the matrix $\mathcal{R} = A + BH(b)$ is not singular. It results that

$$\mathbf{x}^u(t) = H(t)\mathcal{R}^{-1}\gamma + \int_a^b K(t,s)u(s)\mathrm{d}s, \tag{12}$$

where $K(t,s) = \varphi_+(t,s) - H(t)\mathcal{R}^{-1}B\varphi(b,s)$ and

$$\varphi_+(t,s) = \begin{cases} \varphi(t,s), & \text{if} \quad a \le s \le t \le b \\ 0, & \text{if} \quad a \le t < s \le b. \end{cases}$$

To solve the OCP (8)–(10) by the gradient method, it requires to construct the sequence

$$u_{k+1} = u_k - \mu_k I'(u_k)$$

starting with a function $u_0 \in L_2[a,b]$. The descent parameter μ_k is usually computed as the solution of the one dimensional optimization problem

$$I(u_k - \mu_k I'(u_k)) = \min\{I(u_k - \mu I'(u_k)) : \mu \ge 0\}.$$

If we denote $L(\mathbf{x},u,t)=[u(t)-f(x_1(t),x_2(t),\ldots,x_m(t),t)]^2$ then the Gâteaux derivate of the cost functional is

$$I'(u)(\delta u)=\int_a^b\Big[\langle L_x(\mathbf{x}^u(t),u(t),t),\delta\mathbf{x}(t)\rangle+L_u(\mathbf{x}^u(t),u(t),t)\delta u(t)\Big]\mathrm{d}t,$$

where the functions $\delta\mathbf{x}$ and δu satisfy the linear boundary value problem

$$\begin{aligned}\dot{\delta\mathbf{x}}(t)&=Q\,\delta\mathbf{x}(t)+\xi_m\,\delta u(t),\qquad t\in[a,b],\\ A\,\delta\mathbf{x}(a)+B\,\delta\mathbf{x}(b)&=0.\end{aligned}$$

From (12) it results that

$$\delta\mathbf{x}(t)=\int_a^b K(t,s)\delta u(s)\mathrm{d}s$$

and then

$$\begin{aligned}&I'(u)(\delta u)=\\&=\int_a^b\left[\int_a^b\Big\langle L_x(\mathbf{x}^u(t),u(t),t),K(t,s)\Big\rangle\mathrm{d}t+L_u(\mathbf{x}^u(s),u(s),s)\right]\delta u(s)\mathrm{d}s.\end{aligned}$$

Hence the expression of the gradient becomes

$$I'(u)(s)=\int_a^b\Big\langle L_x(\mathbf{x}^u(t),u(t),t),K(t,s)\Big\rangle\mathrm{d}t+L_u(\mathbf{x}^u(s),u(s),s).$$

For the problem (3)–(5) the gradient of the cost functional may be computed by

$$I'(u)=L_u(x_1^u,x_2^u,u,t)-p_2^u, \tag{13}$$

where p_1^u and p_2^u are the solutions of the following two point boundary value problem (the co-state system)

$$\begin{aligned}\dot{p}_1&=L_{x_1}(x_1^u,x_2^u,u,t), &(14)\\ \dot{p}_2&=-p_1+L_{x_2}(x_1^u,x_2^u,u,t), &(15)\\ p_2(0)&=0, &(16)\\ p_2(T)&=0. &(17)\end{aligned}$$

In this case, for the control function u the corresponding trajectory is given by

$$x_1^u(t)=\alpha+\tfrac{\beta-\alpha}{T}t+\int_0^t(t-s)u(s)\mathrm{d}s-\tfrac{t}{T}\int_0^T(T-s)u(s)\mathrm{d}s, \tag{18}$$

$$x_2^u(t)=\tfrac{\beta-\alpha}{T}+\int_0^t u(s)\mathrm{d}s-\tfrac{1}{T}\int_0^T(T-s)u(s)\mathrm{d}s. \tag{19}$$

From (14)–(17) it follows that

(20)

$$
\begin{aligned}
&p_2^u(t) = \\
&= \int_0^t \Big[L_{x_2}(x_1^u(s), x_2^u(s), u(s), s) - (t-s)L_{x_1}(x_1^u(s), x_2^u(s), u(s), s)\Big]\mathrm{d}s - \\
&\quad - \tfrac{t}{T}\int_0^T \Big[L_{x_2}(x_1^u(s), x_2^u(s), u(s), s) - (T-s)L_{x_1}(x_1^u(s), x_2^u(s), u(s), s)\Big]\mathrm{d}s.
\end{aligned}
$$

3. THE CONVERGENCE RESULT

We state a convergence result for the method considered above applied to the problem (3)–(5).

If $u_0 \in L_2[0,T]$ we denote by $M_{I(u_0)}$ the set defined by

$$M_{I(u_0)} = \{u \in L_2[0,T] : I(u) \le I(u_0)\}$$

and we introduce the assumption:

(H) For any $u, v \in M_{I(u_0)}$ there exists $C > 0$ such that

$$\big|f(x_1^u, x_2^u, t) - f(x_1^v, x_2^v, t)\big| \le C\|u - v\|_2;$$

$$
\begin{aligned}
&\big|u\tfrac{\partial f}{\partial x_k}(x_1^u, x_2^u, t) - v\tfrac{\partial f}{\partial x_k}(x_1^v, x_2^v, t)\big| \le \\
&\le C\big[|u(t) - v(t)| + |x_1^u(t) - x_1^v(t)| + |x_2^u(t) - x_2^v(t)|\big];
\end{aligned}
$$

$$
\begin{aligned}
&\big|f(x_1^u, x_2^u, t)\tfrac{\partial f}{\partial x_k}(x_1^u, x_2^u, t) - f(x_1^v, x_2^v, t)\tfrac{\partial f}{\partial x_k}(x_1^v, x_2^v, t)\big| \le \\
&\le C\big[|x_1^u(t) - x_1^v(t)| + |x_2^u() - x_2^v(t)|\big]
\end{aligned}
$$

for any $t \in [0,T]$ and $k \in \{1,2\}$.

If the function f has continuous and bounded first and second order partial derivatives then the assumption (H) is satisfied.

We state some preliminary results.

Theorem 1. *There exist the positive constants C_1 and C_2 such that*

$$
\begin{aligned}
|x_1^u(t) - x_1^v(t)| \le C_1\|u - v\|_2, \\
|x_2^u(t) - x_2^v(t)| \le C_2\|u - v\|_2,
\end{aligned}
$$

for any $u, v \in L_2[0,T]$ and any $t \in [0,T]$.

Proof. From (18) we find

$$x_1^u(t) - x_1^v(t) = \int_0^t (t-s)[u(s)-v(s)]\mathrm{d}s - \tfrac{t}{T}\int_0^T (T-s)[u(s)-v(s)]\mathrm{d}s.$$

It follows that

$$\begin{aligned}|x_1^u(t) - x_1^v(t)| \leq & \Big(\int_0^t (t-s)^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^t [u(s)-v(s)]^2\mathrm{d}s\Big)^{\frac{1}{2}} + \\ & + \Big(\int_0^T (T-s)^2\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^T [u(s)-v(s)]^2\mathrm{d}s\Big)^{\frac{1}{2}} \\ \leq & \tfrac{2\sqrt{3}}{3} T^{\frac{3}{2}} \|u-v\|_2.\end{aligned}$$

Thus $C_1 = \frac{2\sqrt{3}}{3}T^{\frac{3}{2}}\|u-v\|_2$.

Analogously, we deduce

$$|x_2^u(t) - x_2^v(t)| \leq C_2\|u-v\|_2,$$

with $C_2 = (1+\frac{\sqrt{3}}{3})T^{\frac{1}{2}}$. □

THEOREM 2. *If the assumption* (H) *is valid, then there exists the positive constants C_3 and C_4 such that*

$$\begin{aligned}|p_2^u(t) - p_2^v(t)| &\leq C_3\|u-v\|_2, \qquad \forall t \in [0,T],\\ \|I'(u) - I'(v)\|_2 &\leq C_4\|u-v\|_2,\end{aligned}$$

for any $u, v \in M_{I(u_0)}$.

Proof. (i) First, from

$$\begin{aligned}& L_{x_k}(x_1^u, x_2^u, u, t) - L_{x_k}(x_1^v, x_2^v, v, t) = \\ &= -2\big[u - f(x_1^u, x_2^u, t)\big]\frac{\partial f}{\partial x_k}(x_1^u, x_2^u, t) + 2\big[v - f(x_1^v, x_2^v, t)\big]\frac{\partial f}{\partial x_k}(x_1^v, x_2^v, t),\end{aligned}$$

using the assumption (H) we deduce that

$$\begin{aligned}& \big|L_{x_k}(x_1^u, x_2^u, u, t) - L_{x_k}(x_1^v, x_2^v, v, t)\big| \leq \\ & \leq 2C|u(t)-v(t)| + 4C\big(|x_1^u(t) - x_1^v(t)| + |x_2^u(t) - x_2^v(t)|\big) \\ & \leq 2C|u(t)-v(t)| + 4C(C_1+C_2)\|u-v\|_2.\end{aligned}$$

Then, from

$$\begin{aligned}& p_2^u(t) - p_2^v(t) = \\ &= \int_0^t \Big\{\big[L_{x_2}(x_1^u(s), x_2^u(s), u(s), s) - L_{x_2}(x_1^v(s), x_2^v(s), v(s), s)\big] - \\ &\quad - (t-s)\big[L_{x_1}(x_1^u(s), x_1^u(s), u(s), s) - L_{x_1}(x_1^v(s), x_1^v(s), v(s), s)\big]\Big\}\mathrm{d}s - \\ &\quad - \frac{t}{T}\int_0^T \Big\{\big[L_{x_2}(x_1^u(s), x_2^u(s), u(s), s) - L_{x_2}(x_1^v(s), x_2^v(s), v(s), s)\big] - \\ &\quad - (T-s)\big[L_{x_1}(x_1^u(s), x_1^u(s), u(s), s) - L_{x_1}(x_1^v(s), x_1^v(s), v(s), s)\big]\Big\}\mathrm{d}s,\end{aligned}$$

using the above inequalities we have

$$\begin{aligned}
&|p_2^u(t) - p_2^v(t)| \le \\
&\le 2\int_0^T \big[2C|u(s)-v(s)| + 4C(C_1+C_2)\|u-v\|_2\big]\mathrm{d}s + \\
&\quad + \tfrac{2\sqrt{3}}{3}T^{\frac{3}{2}}\Big(\int_0^T \big[2C|u(s)-v(s)| + 4C(C_1+C_2)\|u-v\|_2\big]^2\mathrm{d}s\Big)^{\frac{1}{2}} \\
&\le C_3\|u-v\|_2,
\end{aligned}$$

with $C_3 = 4C\sqrt{T} + 8C(C_1+C_2) + \frac{4\sqrt{3}}{3}T^{\frac{3}{2}}C\sqrt{2+8(C_1+C_2)}$.

(ii) From the equality

$$\begin{aligned}
&I'(u)(t) - I'(v)(t) = \\
&= \big[L_u(x_1^u, x_2^u, u, t) - p_2^u(t)\big] - \big[L_u(x_1^v, x_2^v, v, t) - p_2^v(t)\big] \\
&= 2\big[u(t)-v(t)\big] - 2\big[f(x_1^u, x_2^u, t) - f(x_1^v, x_2^v, t)\big] - \big[p_2^u(t) - p_2^v(t)\big]
\end{aligned}$$

we obtain

$$\big|I'(u)(t) - I'(v)(t)\big| \le 2|u(t)-v(t)| + (2C+C_3)\|u-v\|_2.$$

Hence

$$\begin{aligned}
\|I'(u) - I'(v)\| &= \Big(\int_0^T \big|I'(u)(t) - I'(v)(t)\big|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\le \Big(\int_0^T \Big[2\big|u(t)-v(t)\big| + (2C+C_3)\|u-v\|_2\Big]^2\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\le C_4\|u-v\|_2
\end{aligned}$$

and $C_4 = 2 + (2C+C_3)T$. □

Let U be a Hilbert space and $J : U \to \mathbb{R}$ a Gâteaux differentiable functional. We shall establish an adequate convergence theorem for the gradient method used to solve the optimization problem

$$\min_{u\in U} J(u).$$

THEOREM 3. *Let $u^0 \in U$. If*

(1) *J is Gâteaux differentiable and bounded below;*

(2) *There exists $L > 0$ such that*

$$\|J'(u) - J'(v)\| \le L\|u-v\|,$$

for any $u, v \in M_{J(u^0)} = \{u \in U : J(u) \le J(u^0)\}$;

then there exists $\delta \in (0, \frac{1}{L})$ such that the sequence $(u^k)_{k\in N}$, defined by

$$u^{k+1} = u^k - \mu_k J'(u^k), \quad \textit{with} \quad \mu_k \in E_\delta = [\delta, \tfrac{2}{L} - \delta],$$

has the properties:

a) *The sequence* $(J(u^k))_{k\in N}$ *is convergent;*

b) $\lim_{k\to\infty} J'(u^k) = 0$.

Proof. First we prove that there exists $\delta \in (0, \frac{1}{L})$ such that for any $u \in M_{J(u^0)}$, for any $\mu \in E_\delta$ and for any $t \in [0, \mu]$ we have $u - tJ'(u) \in M_{J(u)}$.

Let us suppose, by contrary, that for any $\delta \in (0, \frac{1}{L})$ there exists $u_1 \in M_{J(u^0)}, \mu_1 \in E_\delta$ and $t_1 \in [0, \mu_1]$ such that

$$u_1 - t_1 J'(u_1) \notin M_{J(u_1)} \Leftrightarrow J(u_1 - t_1 J'(u_1)) > J(u_1).$$

Obviously $J'(u_1) \neq 0$. From

$$\lim_{\lambda\to 0} \tfrac{1}{\lambda}\left[J(u_1 - \lambda J'(u_1)) - J(u_1)\right] = -\|J'(u_1)\|^2 < 0$$

it follows that there exists μ_2 such that for any $\mu \in [0, \mu_2] J(u_1 - \mu J'(u_1)) < J(u_1)$. Necessarily $\mu_2 < t_1$. The continuity of the function $t \mapsto J(u_1 - tJ'(u_1))$ implies that there exists $t_2 \in [\mu_2, t_1]$ such that $J(u_1 - t_2 J'(u_1)) = J(u_1)$ and for any $t \in [0, t_2]$, $u_1 - tJ'(u_1) \in M_{J(u_1)}$.

The following relations are then valid

$$0 = J(u_1 - t_2 J'(u_1)) - J(u_1) \leq (\tfrac{Lt_2^2}{2} - t_2)\|J'(u_1)\| < 0,$$

which are contradictory.

Consequently, the assertions of the theorem follows from the inequalities

$$J(u^{k+1}) - J(u^k) \leq (\tfrac{L\mu_k^2}{2} - \mu_k)\|J'(u^k)\| \leq (\tfrac{L\delta^2}{2} - \delta)\|J'(u^k)\|, \qquad \forall k \in \mathbb{N}. \quad \square$$

Because the functional I is Gâteaux differentiable, bounded below and satisfies the Lipschitz property (Theorem 3.2), as a consequence of the Theorem 3.3 we obtain the following result.

THEOREM 4. *If the hypothesis* (H) *is valid then the sequence* $(u^k)_{k\in N}$ *constructed by the gradient method* (17) *to solve the NTPBVP* (1)–(3) *has the following properties:*

(1) *The sequence* $(I(u^k))_{k\in N}$ *is convergent;*

(2) $\lim_{k\to\infty} I'(u^k) = 0$.

4. IMPLEMENTATION OF THE METHOD

Let $n \in N^*$. On $[0, 1]$ we consider the mesh $0 = t_0 < t_1 < \ldots < t_n = 1$ where $t_i = ih, i = 0, 1, \ldots, n$ and $h = 1/n$. Let

$$\begin{aligned} u_h^k &= (u_0^k, u_1^k, \ldots, u_n^k), \\ x_{1,h}^k &= (x_{1,0}^k, x_{1,1}^k, \ldots, x_{1,n}^k), \\ x_{2,h}^k &= (x_{2,0}^k, x_{2,1}^k, \ldots, x_{2,n}^k), \\ p_{2,h}^k &= (p_{2,0}^k, p_{2,1}^k, \ldots, p_{2,n}^k) \end{aligned}$$

be the discretization of the functions $u^k, x_1^{u^k}, x_2^{u^k}$ and $p_2^{u^k}$ respectively, at the points $t_i, i = 0, 1, \dots, n$.

Using the formulas (18), (19), (20) and (6) $x_{1,h}^k, x_{2,h}^k, p_{2,h}^k$ and $I(u_h^k)$ were computed with the trapezoidal rule of integration.

If $s_h^k = (s_0^k, s_1^k, \dots, s_n^k)$ are defined by

$$-s_i^k = 2\big[u_i^k - f(x_{1,i}^k, x_{2,i}^k, t_i)\big] - p_{2,i}^k \quad i = 0, 1, \dots, n,$$

then using an algorithm of one dimensional optimization based on a parabolic interpolation, it is find μ_k as

$$I(u_h^k + \mu_k s_h^k) = \min\big\{I(u_h^k + \mu s_h^k) : \mu \geq 0\big\}.$$

The next approximation is given by

$$u_i^{k+1} = u_i^k + \mu_k s_i^k \quad i = 0, 1, \dots, n.$$

The stopping condition is given by

$$\Big(\sum_{i=0}^{n}(u_i^{k+1} - u_i^k)^2\Big)^{1/2} < \epsilon = 0.001.$$

5. NUMERICAL EXAMPLES

Example 1. Consider the equation

$$\ddot{x} = \exp x, \qquad x(0) = x(1) = 0$$

with the solution

$$x(t) = \ln 2 + 2\ln\big(c/\cos\tfrac{c(t-0.5)}{2}\big),$$

where $c \approx 1.3360656$. In this case $f(x_1, x_2, t) = e^{x_1}$.

The results are presented in Table 1. On the other hand, the value of the cost functional $I(u_h^k)$ and the error

$$e_k = \Big(\sum_{i=0}^{n}[x_{1,i}^k - x(t_i)]^2\Big)^{1/2}$$

are presented in Table 2. □

Example 2. Consider the equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big[(x^2 + 0.1)\frac{\mathrm{d}x}{\mathrm{d}t}\Big] + x = 10t^4 - 20t^3 + 11t^2 + 0.2, \qquad x(0) = x(1) = 0$$

with the solution $x(t) = t - t^2$ (Sokolowski J., Matsumura T., Sakawa Y., [12]). In this case

$$f(x_1, x_2, t) = \frac{x_1 - 2x_1x_2^2 - 10t^4 + 20t^3 - 11t^2 + t - 0.2}{x_1^2 + 0.1}.$$

The results are presented in Table 3 and Table 4, respectively. □

Remark. The discretization was done with $n = 10 (h = 0.1)$. The initial approximations were taken $u_i^0 = 0$, $i = 0, 1, \dots, n$. □

Table 1. The discrete solution.

t_j	$x_{1,j}^k$	$x_1(t_j)$	$\lvert x_{1,j}^k - x_1(t_j)\rvert$
0	0	0	0
.1	−.0414	−.0414	.2750E−4
.2	−.0732	−.0732	.5038E−4
.3	−.0957	−.0958	.6596E−4
.4	−.1092	−.1092	.7502E−4
.5	−.1136	−.1137	.7799E−4
.6	−.1092	−.1092	.7502E−4
.7	−.0957	−.0958	.6596E−4
.8	−.0732	−.0732	.5038E−4
.9	−.0414	−.0414	.2750E−4
1	0	0	0

Table 2. The evolution of the cost functional.

k	$I(u^k)$	e_k
1	1.0000000	.26327643E+0
2	.46221580E−2	.79034139E−2
3	.19229175E−4	.13256727E−2
4	.89363358E−7	.14495543E−3
5	.41923597E−9	.18614450E−3

Table 3. The discrete solution.

t_j	$x_{1,j}^k$	$x_1(t_j)$	$\lvert x_{1,j}^k - x_1(t_j)\rvert$
0	.0000	.0000	0
.1	.0899	.0900	.9783E−5
.2	.1600	.1600	.1367E−4
.3	.2100	.2100	.1578E−4
.4	.2400	.2400	.1712E−4
.5	.2500	.2500	.1761E−4
.6	.2400	.2400	.1712E−4
.7	.2100	.2100	.1578E−4
.8	.1600	.1600	.1367E−4
.9	.0899	.0900	.9783E−5
1	.0000	.0000	0

Table 4. The evolution of the cost functional.

k	$I(u^k)$	e_k
1	.15649330E+2	.57732140E+0
2	.67351863E+0	.23050035E−1
3	.22708556E+0	.68533936E−1
4	.84461831E−1	.84342953E−2
5	.30519258E−1	.26329220E−1
6	.11536148E−1	.33611155E−2
7	.43714834E−2	.10240440E−1
8	.16967369E−2	.14039690E−2
9	.66307694E−3	.40626219E−2
10	.26265512E−3	.58864895E−3
11	.10459776E−3	.16319149E−2
12	.41961555E−4	.24410568E−3
13	.16892649E−4	.65968634E−3
14	.68264829E−5	.10047550E−3
15	.27639249E−5	.26765030E−3
16	.11212327E−5	.41146389E−4
17	.45530150E−6	.10879322E−3
18	.18506415E−6	.16800688E−4

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