REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 31 (2002) no. 2, pp. 123–133
ictp.acad.ro/jnaat

ON THE RATE OF CONVERGENCE OF BASKAKOV-KANTOROVICH-BÉZIER OPERATORS FOR BOUNDED VARIATION FUNCTIONS

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Abstract. In the present paper we introduce Baskakov-Kantorovich-Bézier operators and study their rate of convergence for functions of bounded variation.

MSC 2000. 41A36, 41A25, 26A45.

Keywords. Approximation by positive operators, rate of convergence, degree of approximation, functions of bounded variation, total variation.

1. INTRODUCTION

Let $W(0,\infty)$ be the class of functions f which are locally integrable on $(0,\infty)$ and are of polynomial growth as $t\to\infty$, i.e., for some positive r, there holds $f(t)=\mathcal{O}(t^r)$, as $t\to\infty$. We consider the Kantorovich variant $\widehat{V}_n$ of the Baskakov operators associating to each function $f\in W(0,\infty)$ the series

$$\widehat{V}_n(f;x)=n\sum_{k=0}^{\infty}v_{n+1,k}(x)\int_{I_k}f(t)\,\mathrm{d}t,\qquad x\in[0,\infty),\tag{1}$$

where $I_k=[k/n,(k+1)/n]$ and

$$v_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k}.$$

Note that the operators (1) are well defined, for sufficiently large n, provided $f\in W(0,\infty)$.

We mention a slightly different definition for the Kantorovich variant V_n^* of the Baskakov operators, given by

$$V_n^*(f;x)=n\sum_{k=0}^{\infty}v_{n,k}(x)\int_{I_k}f(t)\,\mathrm{d}t,\qquad x\in[0,\infty),\tag{2}$$

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(see, e.g., [3, Eq. (9.2.3), p. 115]). The former definition (1) has the advantage to satisfy the relation

$$\tag{3} \tfrac{\mathrm{d}}{\mathrm{d}x} V_n(F;x) = \widehat{V}_n(f;x),$$

where $F = \int f$ is a primitive of f and V_n denotes the ordinary Baskakov operators given by

$$V_n(f;x) = \sum_{k=0}^{\infty} v_{n,k}\left(x\right) f\left(\tfrac{k}{n}\right).$$

In the present paper we introduce the Bézier variant of the operators (1). For each function $f \in W\left(0,\infty\right)$ and $\alpha \geq 1$, we introduce the Bézier type Baskakov-Kantorovich operators $\widehat{V}_{n,\alpha}$ as

$$\tag{4} \widehat{V}_{n,\alpha}(f;x) = n \sum_{k=0}^{\infty} Q_{n+1,k}^{(\alpha)}\left(x\right) \int_{I_k} f\left(t\right) \mathrm{d}t,$$

where

$$Q_{n,k}^{(\alpha)}\left(x\right) = J_{n,k}^{\alpha}\left(x\right) - J_{n,k+1}^{\alpha}\left(x\right)$$

and

$$J_{n,k}\left(x\right) = \sum_{j=k}^{\infty} v_{n,j}\left(x\right)$$

is the Baskakov-Bézier basis function. It is obvious that $\widehat{V}_{n,\alpha}$ are positive linear operators and $\widehat{V}_{n,\alpha}(1;x) = 1$. In the special case $\alpha = 1$, the operators $\widehat{V}_{n,\alpha}$ reduce to the operators $\widehat{V}_n \equiv \widehat{V}_{n,1}$. Some basic properties of $J_{n,k}$ are as follows:

(i) $J_{n,k}\left(x\right) - J_{n,k+1}\left(x\right) = v_{n,k}\left(x\right), \qquad k = 0,1,2,\ldots;$

(ii) $J_{n,k}'\left(x\right) = \left(n+1\right) v_{n+1,k-1}\left(x\right), \qquad k = 1,2,3,\ldots;$

(iii) $J_{n,k}\left(x\right) = \left(n+1\right) \int_0^x v_{n+1,k-1}\left(t\right) \mathrm{d}t, \qquad k = 1,2,3,\ldots;$

(iv) $0 < \ldots < J_{n,k+1}\left(x\right) < J_{n,k}\left(x\right) < \ldots < J_{n,1}\left(x\right) < J_{n,0}\left(x\right) \equiv 1, \quad x > 0;$

(v) $J_{n,k}$ is strictly increasing on $[0,\infty)$.

Rates of convergence on functions of bounded variation, for different Bézier type operators, were studied in several papers, e.g., [6], [7], [8], [1]. In the present paper we estimate the rate of convergence by the Baskakov-Kantorovich-Bézier operators (4).

Furthermore, we find the limit of the sequence $\widehat{V}_{n,\alpha}(f;x)$ for bounded locally integrable functions f having a discontinuity of the first kind in $x \in (0,\infty)$.

2. THE MAIN RESULTS

As main result we derive the following estimate on the rate of convergence.

THEOREM 1. *Assume that $f \in W(0,\infty)$ is a function of bounded variation on every finite subinterval of $(0,\infty)$. Furthermore, let $\alpha \geq 1$, $x \in (0,\infty)$ and $\lambda > 1$ be given. Then, for each $r \in \mathbb{N}$, there exists a constant $M(f,\alpha,r,x)$, such that, for sufficiently large n, the Baskakov-Kantorovich-Bézier operators $\widehat{V}_{n,\alpha}$ satisfy the estimate*

$$\left|\widehat{V}_{n,\alpha}(f;x) - \left[\tfrac{1}{2^\alpha} f(x+) + \left(1 - \tfrac{1}{2^\alpha}\right) f(x-)\right]\right| \leq \tag{5}$$

$$\leq \frac{2\alpha\lambda(1+x)+x}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + \frac{7\alpha\sqrt{1+x}}{2\sqrt{(n+1)x}} \left| f(x+) - f(x-) \right| + \frac{M(f,\alpha,r,x)}{n^r},$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-), & 0 \leq t < x, \\ 0, & t = x, \\ f(t) - f(x+), & x < t < \infty, \end{cases} \tag{6}$$

and $\bigvee_a^b (g_x)$ is the total variation of g_x on $[a,b]$.

REMARK 1. The exponent r in the $\mathcal{O}$–term of Eq. (5) can be chosen arbitrary large. □

As an immediate consequence of Theorem 1 we obtain in the special case $\alpha = 1$ the following estimate for the Baskakov-Kantorovich operators $\widehat{V}_n$.

COROLLARY 2. *Under the assumptions of Theorem 1 there holds, for sufficiently large n,*

$$\left|\widehat{V}_n(f;x) - \tfrac{1}{2}\left[f(x+) + f(x-)\right]\right| \leq$$

$$\leq \frac{2\lambda(1+x)+x}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + \frac{7\sqrt{1+x}}{2\sqrt{(n+1)x}} \left| f(x+) - f(x-) \right| + \frac{M(f,1,r,x)}{n^r},$$

where g_x is as defined in Theorem 1.

THEOREM 3. *Let $x \in (0,\infty)$. If $f \in L(0,\infty)$ has a discontinuity of the first kind in x, then we have*

$$\lim_{n\to\infty} \widehat{V}_{n,\alpha}(f;x) = \tfrac{1}{2^\alpha} f(x+) + \left(1 - \tfrac{1}{2^\alpha}\right) f(x-).$$

3. AUXILIARY RESULTS

In order to prove our main result we shall need the following lemmas. Throughout the paper let e_r denote the monomials $e_r(t) = t^r$, $r = 0,1,2,\ldots$, and, for each real x, put $\psi_x(t) = t - x$.

LEMMA 4. [9, Lm. 1]. *For all $x>0$ and $n,k\in\mathbb{N}$, there holds*

$$Q_{n,k}^{(\alpha)}(x)\leq\alpha\ v_{n,k}(x)<\alpha\sqrt{\tfrac{1+x}{2enx}}.$$

LEMMA 5. [2, Cor. 3]. *For each fixed $x\in[0,\infty)$ and $m\in\mathbb{N}_0$, the central moments $\widehat{V}_n(\psi_x^m;x)$ of the Baskakov-Kantorovich operators* (1) *satisfy*

$$\widehat{V}_n(\psi_x^m;x)=\mathcal{O}\big(n^{-\lfloor(m+1)/2\rfloor}\big),\qquad \text{as } n\to\infty.$$

In particular, we have

$$\begin{aligned}\widehat{V}_n(e_0;x)&=1,\\ \widehat{V}_n(e_1;x)&=x+\tfrac{1+2x}{2n},\\ \widehat{V}_n(e_2;x)&=x^2+\tfrac{3x^2(3n+2)+6(n+1)x+1}{3n^2},\\ \widehat{V}_n(\psi_x^2;x)&=\tfrac{3x(1+x)(n+2)+1}{3n^2}.\end{aligned}$$

REMARK 2. Note that, given any $\lambda>1$ and any $x>0$, for all n sufficiently large, we have the estimate

$$\widehat{V}_n(\psi_x^2;x)<\tfrac{\lambda x(1+x)}{n}.\qquad\square$$

Throughout the paper let

$$K_{n,\alpha}(x,t)=n\sum_{k=0}^{\infty}Q_{n+1,k}^{(\alpha)}(x)\ \chi_{n,k}(t),$$

where $\chi_{n,k}$ denotes the characteristic function of the interval $[k/n,(k+1)/n]$ with respect to $[0,\infty)$. Given a function $f\in W(0,\infty)$, with this definition there holds, for all sufficiently large n,

$$\widehat{V}_{n,\alpha}(f;x)=\int_0^\infty K_{n,\alpha}(x,t)\ f(t)\,\mathrm{d}t. \tag{7}$$

Furthermore, put

$$\lambda_{n,\alpha}(x,y)=\int_0^y K_{n,\alpha}(x,t)\,\mathrm{d}t. \tag{8}$$

Note that, in particular,

$$\lambda_{n,\alpha}(x,\infty)=\int_0^\infty K_{n,\alpha}(x,u)\,\mathrm{d}u=1.$$

LEMMA 6. *Let $x\in(0,\infty)$. For each $\lambda>1$, and for all sufficiently large n, we have:*

$$\lambda_{n,\alpha}(x,y)=\int_0^y K_{n,\alpha}(x,t)\,\mathrm{d}t\leq\tfrac{\lambda\alpha x(1+x)}{n(x-y)^2},\qquad 0\leq y<x, \tag{9}$$

$$1-\lambda_{n,\alpha}(x,z)=\int_z^\infty K_{n,\alpha}(x,t)\,\mathrm{d}t\leq\tfrac{\lambda\alpha x(1+x)}{n(z-x)^2},\qquad x<z<\infty. \tag{10}$$

Proof. We first prove Eq. (9). There holds

$$\begin{aligned}\int_0^y K_{n,\alpha}(x,t)\,\mathrm{d}t &\le \int_0^y K_{n,\alpha}(x,t)\,\tfrac{(x-t)^2}{(x-y)^2}\mathrm{d}t\\ &\le (x-y)^{-2}\,\widehat{V}_{n,\alpha}(\psi_x^2;x)\\ &\le \alpha\,(x-y)^{-2}\,\widehat{V}_{n,1}(\psi_x^2;x),\end{aligned}$$

where we applied Lemma 4. Now Eq. (9) is a consequence of Remark 2. The proof of Eq. (10) is similar. □

The following lemma is the well-known Berry-Esseen bound for the central limit theorem of probability theory. It can be used to estimate upper and lower bounds for the partial sums of Baskakov basis functions.

Lemma 7. (Berry-Esseen) [4, p. 300], [5, p. 342]. *Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of independent and identically distributed random variables with the expectation $E(\xi_1)=a_1$, the variance $E(\xi_1-a_1)^2=\sigma^2>0$, $E|\xi_1-a_1|^3=\rho<\infty$, and let F_n stand for the distribution function of $\sum_{k=1}^{n}(\xi_k-a_1)/(\sigma\sqrt{n})$. Then there exists an absolute constant C, $1/\sqrt{2\pi}\le C<0.82$, such that for all t and n, there holds*

$$\left|F_n(t)-\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-u^2/2}\mathrm{d}u\right|\le\frac{C\rho}{\sigma^3\sqrt{n}}.$$

Lemma 8. [9, Lm. 3]. *Let $(\xi_k)_{k=1}^{\infty}$ be a sequence of independent random variables with the same geometric distribution*

$$P(\xi_1=k)=\tfrac{1}{1+x}\left(\tfrac{x}{1+x}\right)^k,\qquad k=1,2,3,\ldots,$$

where $x>0$ is a parameter. Then there holds

$$\begin{aligned}E(\xi_1)&=x,\\ E(\xi_1-E\xi_1)^2&=x^2+x,\\ E|\xi_1-E\xi_1|^3&\le 3x(1+x)^2.\end{aligned}$$

Lemma 9. *For all $x\in(0,\infty)$, there holds*

$$\Big|\sum_{k>nx}v_{n+1,k}(x)-\tfrac12\Big|\le\tfrac{3\sqrt{1+x}}{\sqrt{(n+1)x}}.$$

Proof. We follow the proof of [9, Lm. 5]. Let $(\xi_k)_{k=1}^{\infty}$ be the sequence of independent random variables as defined in Lemma 8. Then the probability distribution of the random variable $\eta_n=\sum_{k=1}^{n}\xi_k$ is

$$P(\eta_n=k)=\binom{n+k-1}{k}\tfrac{x^k}{(1+x)^{n+k}}=v_{n,k}(x),\quad k=1,2,3,\ldots.$$

Therefore, we have

$$\begin{aligned}\sum_{k>(n-1)x} v_{n,k}(x) &= P(\eta_n > (n-1)x) \\ &= 1 - P(\eta_n - nx \le -x) \\ &= 1 - F_n\Big(\tfrac{-x}{\sqrt{nx(1+x)}}\Big).\end{aligned}$$

Application of Lemma 7 in combination with Lemma 8 implies

$$\begin{aligned}&\Big|\sum_{k>(n-1)x} v_{n,k}(x) - \tfrac{1}{2}\Big| = \\ &= \Big|F_n\left(\tfrac{-x}{\sigma\sqrt{n}}\right) - \tfrac{1}{2}\Big| \\ &= \Big|F_n\left(\tfrac{-x}{\sigma\sqrt{n}}\right) - \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-t^2/2}\mathrm{d}t\Big| \\ &\le \Big|F_n\left(\tfrac{-x}{\sigma\sqrt{n}}\right) - \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x/(\sigma\sqrt{n})} e^{-t^2/2}\mathrm{d}t\Big| + \tfrac{1}{\sqrt{2\pi}}\int_{-x/(\sigma\sqrt{n})}^{0} e^{-t^2/2}\mathrm{d}t \\ &\le \tfrac{C\rho}{\sigma^3\sqrt{n}} + \tfrac{x}{\sqrt{2\pi n}\sigma} \\ &\le \tfrac{3\cdot 0.82\sqrt{1+x}}{\sqrt{nx}} + \tfrac{x}{\sqrt{2\pi nx(1+x)}} \\ &< \tfrac{2.5(1+x)}{\sqrt{nx(1+x)}} + \tfrac{0.4x}{\sqrt{nx(1+x)}} \\ &< \tfrac{3\sqrt{1+x}}{\sqrt{nx}}.\end{aligned}$$

Replacing n by $n+1$ completes the proof. □

4. PROOF OF THE MAIN RESULTS

Proof of Theorem 1. Our starting-point is the identity

$$\begin{aligned}f(t) &= \tfrac{1}{2^\alpha} f(x+) + \Big(1 - \tfrac{1}{2^\alpha}\Big) f(x-) + \tfrac{f(x+)-f(x-)}{2^\alpha}\operatorname{sign}_x(t) \\ &\quad + g_x(t) + \delta_x(t)\Big[f(x) - \tfrac{1}{2^\alpha} f(x+) - \Big(1 - \tfrac{1}{2^\alpha}\Big) f(x-)\Big],\end{aligned}$$

where

$$\operatorname{sign}_x(t) = \begin{cases} 2^\alpha - 1, & t > x, \\ 0, & t = x, \\ -1, & t < x, \end{cases}$$

$\delta_x(t) = 1$, $t = x$, and $\delta_x(t) = 0$, $t \ne x$. Since $\widehat{V}_{n,\alpha}(\delta_x; x) = 0$, we conclude

$$\begin{aligned}&\Big|\widehat{V}_{n,\alpha}(f;x) - \Big[\tfrac{1}{2^\alpha} f(x+) + \Big(1 - \tfrac{1}{2^\alpha}\Big) f(x-)\Big]\Big| \le \\ &\le \tfrac{1}{2^\alpha}\left|f(x+) - f(x-)\right| \Big|\widehat{V}_{n,\alpha}(\operatorname{sign}_x(t); x)\Big| + \Big|\widehat{V}_{n,\alpha}(g_x; x)\Big|. \qquad (11)\end{aligned}$$

First, we estimate $\widehat{V}_{n,\alpha}(\mathrm{sign}_x(t);x)$ as follows. Choose k' such that $x \in [k'/n,(k'+1)/n)$. Hence,

$$\begin{aligned}\widehat{V}_{n,\alpha}(\mathrm{sign}_x(t);x) &= \sum_{k=0}^{k'-1}(-1)\,Q_{n+1,k}^{(\alpha)}(x) \\ &\quad + nQ_{n+1,k'}^{(\alpha)}(x)\left(\int_{k'/n}^{x}(-1)\,\mathrm{d}t+\int_{x}^{(k'+1)/n}(2^\alpha-1)\,\mathrm{d}t\right) \\ &\quad + \sum_{k=k'+1}^{\infty}(2^\alpha-1)\,Q_{n+1,k}^{(\alpha)}(x) \\ &= 2^\alpha\sum_{k=k'+1}^{\infty}Q_{n+1,k}^{(\alpha)}(x)+nQ_{n+1,k'}^{(\alpha)}(x)\int_{x}^{(k'+1)/n}2^\alpha\mathrm{d}t-1\ ,\end{aligned}$$

since $\sum_{j=0}^{\infty}Q_{n+1,j}^{(\alpha)}(x)=1$. Noting

$$0\le nQ_{n+1,k'}^{(\alpha)}(x)\int_{x}^{(k'+1)/n}2^\alpha\mathrm{d}t\le 2^\alpha\ Q_{n+1,k'}^{(\alpha)}(x)$$

we conclude

$$\begin{aligned}\left|\widehat{V}_{n,\alpha}(\mathrm{sign}_x(t);x)\right| &\le \left|2^\alpha\sum_{k=k'+1}^{\infty}Q_{n+1,k}^{(\alpha)}(x)-1\right|+2^\alpha\ Q_{n+1,k'}^{(\alpha)}(x) \\ &= \left|2^\alpha J_{n+1,k'+1}^{\alpha}(x)-1\right|+2^\alpha\ Q_{n+1,k'}^{(\alpha)}(x)\,.\end{aligned}$$

Application of the inequality $|a^\alpha-b^\alpha|\le\alpha\,|a-b|$, for $0\le a,b\le 1$, and $\alpha\ge 1$, yields

$$\begin{aligned}\left|2^\alpha J_{n+1,k'+1}^{\alpha}(x)-1\right| &\le \alpha 2^\alpha\left|J_{n+1,k'+1}(x)-\tfrac{1}{2}\right| \\ &=\alpha 2^\alpha\left|\sum_{k=k'+1}^{\infty}v_{n+1,k}(x)-\tfrac{1}{2}\right| \\ &=\alpha 2^\alpha\left|\sum_{k>nx}^{\infty}v_{n+1,k}(x)-\tfrac{1}{2}\right|.\end{aligned}$$

Therefore, by Lemma 9 and Lemma 4, we obtain

$$\left|\widehat{V}_{n,\alpha}(\mathrm{sign}_x(t);x)\right|\le\alpha 2^\alpha\tfrac{3\sqrt{1+x}}{\sqrt{(n+1)x}}+2^\alpha\tfrac{\alpha\sqrt{1+x}}{\sqrt{2e(n+1)x}}<\tfrac{7\alpha\cdot 2^{\alpha-1}\sqrt{1+x}}{\sqrt{(n+1)x}}. \tag{12}$$

In order to complete the proof of the theorem we need an estimate of $\widehat{V}_{n,\alpha}(g_x;x)$. We use the integral representation (7) and decompose $[0,\infty)$ into three parts as follows

$$\begin{aligned}\widehat{V}_{n,\alpha}(g_x;x) &= \left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}+\int_{x+x/\sqrt{n}}^{\infty}\right)K_{n,\alpha}(x,t)\ g_x(t)\,\mathrm{d}t \\ &= I_1+I_2+I_3,\ \text{say.}\end{aligned} \tag{13}$$

We start with I_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|g_x(t)| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (g_x)$$

and thus

$$|I_2| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (g_x) \le \tfrac{1}{n} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) . \tag{14}$$

Next we estimate I_1. Put $y = x - x/\sqrt{n}$. Using integration by parts with Eq. (8) we have

$$I_1 = \int_0^y g_x(t) \, \mathrm{d}_t \lambda_{n,\alpha}(x,t) = g_x(y) \lambda_{n,\alpha}(x,y) - \int_0^y \lambda_{n,\alpha}(x,t) \, \mathrm{d}_t g_x(t) .$$

Since $|g_x(y)| = |g_x(y) - g_x(x)| \le \bigvee_y^x (g_x)$, we conclude

$$|I_1| \le \bigvee_y^x (g_x) \;\; \lambda_{n,\alpha}(x,y) + \int_0^y \lambda_{n,\alpha}(x,t) \, \mathrm{d}_t \Big(- \bigvee_t^x (g_x) \Big).$$

Since $y = x - x/\sqrt{n} \le x$, Eq. (9) of Lemma 6 implies, for each $\lambda > 1$ and n sufficiently large,

$$|I_1| \le \tfrac{\alpha\lambda x(1+x)}{n(x-y)^2} \bigvee_y^x (g_x) + \tfrac{\alpha\lambda x(1+x)}{n} \int_0^y \tfrac{1}{(x-t)^2} \mathrm{d}_t \Big(- \bigvee_t^x (g_x) \Big).$$

Integrating the last term by parts, we obtain

$$|I_1| \le \tfrac{\alpha\lambda x(1+x)}{n} \Big(x^{-2} \bigvee_0^x (g_x) + 2 \int_0^y \frac{\bigvee_t^x (g_x)}{(x-t)^3} \mathrm{d}t \Big).$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we get

$$\begin{aligned} \int_0^{x-x/\sqrt{n}} \bigvee_t^x (g_x) (x-t)^{-3} \, \mathrm{d}t &= \sum_{k=1}^{n-1} \int_{x/\sqrt{k+1}}^{x/\sqrt{k}} \bigvee_{x-t}^x (g_x) \, t^{-3} \mathrm{d}t \\ &\le \frac{1}{2x^2} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^x (g_x) . \end{aligned}$$

Hence

$$|I_1| \le \tfrac{2\alpha\lambda(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^x (g_x) . \tag{15}$$

Finally, we estimate I_3. We put

$$\widetilde{g}_x(t) = \begin{cases} g_x(t), & 0 \le t \le 2x, \\ g_x(2x), & 2x < t < \infty, \end{cases}$$

and divide $I_3 = I_{31} + I_{32}$, where

$$I_{31} = \int_{x+x/\sqrt{n}}^{\infty} K_{n,\alpha}(x,t)\ \widetilde{g}_x(t)\,\mathrm{d}t$$

and

$$I_{32} = \int_{2x}^{\infty} K_{n,\alpha}(x,t)\ [g_x(t) - g_x(2x)]\,\mathrm{d}t.$$

With $y = x + x/\sqrt{n}$ the first integral can be written in the form

$$I_{31} = \lim_{R\to+\infty}\Big\{g_x(y)\,[1-\lambda_{n,\alpha}(x,y)] + \widetilde{g}_x(R)\,[\lambda_{n,\alpha}(x,R)-1] + \\ + \int_y^R [1-\lambda_{n,\alpha}(x,t)]\,\mathrm{d}_t\widetilde{g}_x(t)\Big\}.$$

By Eq. (10) of Lemma 6, we conclude, for each $\lambda > 1$ and n sufficiently large,

$$\begin{aligned} |I_{31}| &\le \tfrac{\alpha\lambda x(1+x)}{n}\lim_{R\to+\infty}\Big\{\tfrac{\bigvee_x^y(g_x)}{(y-x)^2} + \tfrac{|\widetilde{g}_x(R)|}{(R-x)^2} + \int_y^R \tfrac{1}{(t-x)^2}\mathrm{d}_t\Big(\bigvee_x^t(\widetilde{g}_x)\Big)\Big\} \\ &= \tfrac{\alpha\lambda x(1+x)}{n}\Big\{\tfrac{\bigvee_x^y(g_x)}{(y-x)^2} + \int_y^{2x}\tfrac{1}{(t-x)^2}\mathrm{d}_t\Big(\bigvee_x^t(g_x)\Big)\Big\}. \end{aligned}$$

In a similar way as above we obtain

$$\int_y^{2x}\tfrac{1}{(t-x)^2}\mathrm{d}_t\Big(\bigvee_x^t(g_x)\Big) \le x^{-2}\bigvee_x^{2x}(g_x) - \frac{\bigvee_x^y(g_x)}{(y-x)^2} + x^{-2}\sum_{k=1}^{n-1}\bigvee_x^{x+x/\sqrt{k}}(g_x)$$

which implies the estimate

$$|I_{31}| \le \tfrac{2\alpha\lambda(1+x)}{nx}\sum_{k=1}^{n}\bigvee_x^{x+x/\sqrt{k}}(g_x)\,. \tag{16}$$

Lastly, we estimate I_{32}. By assumption, there exists an integer r, such that $f(t) = \mathcal{O}(t^{2r})$, as $t\to\infty$. Thus, for a certain constant $M > 0$ depending only on f, x and r, we have

$$\begin{aligned} |I_{32}| &\le Mn\sum_{k=0}^{\infty}Q_{n+1,k}^{(\alpha)}(x)\int_{2x}^{\infty}\chi_{n,k}(t)\,t^{2r}\mathrm{d}t \\ &\le \alpha Mn\sum_{k=0}^{\infty}v_{n+1,k}(x)\int_{2x}^{\infty}\chi_{n,k}(t)\,t^{2r}\mathrm{d}t, \end{aligned}$$

where we used Lemma 4. Obviously, $t \ge 2x$ implies $t \le 2(t-x)$ and it follows

$$|I_{32}| \le 2^{2r}\alpha M\widehat{V}_n(\psi_x^{2r};x).$$

Because the central moments of the Baskakov-Kantorovich operators (1) satisfy $\widehat{V}_n(\psi_x^{2r};x) = \mathcal{O}(n^{-r})$, as $n\to\infty$ [2, Cor. 3], we arrive at

$$I_{32} = \mathcal{O}(n^{-r}), \qquad \text{as } n\to\infty. \tag{17}$$

Collecting the estimates (14), (15), (16), and (17), we obtain with regard to Eq. (13)

$$\left|\widehat{V}_{n,\alpha}(g_x;x)\right| \le \tfrac{2\alpha\lambda(1+x)+x}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + \mathcal{O}\left(n^{-r}\right), \qquad \text{as } n \to \infty. \tag{18}$$

Finally, combining (11), (12), (18), we obtain (5). This completes the proof of Theorem 1. □

Proof of Theorem 3. Since the function ψ_x^2 given by $\psi_x^2(t) = (t-x)^2$ is of bounded variation on every finite subinterval of $[0,\infty)$, we deduce from Theorem 1 that, for all $x \in (0,\infty)$,

$$\lim_{n\to\infty} \widehat{V}_{n,\alpha}(\psi_x^2;x) = 0.$$

If $f \in L_\infty(0,\infty)$, then g_x defined as in (6) is also bounded and is continuous at the point x. By the Korovkin theorem, we conclude

$$\lim_{n\to\infty} \widehat{V}_{n,\alpha}(g_x;x) = g_x(x) = 0.$$

Therefore, the right-hand side of Inequality (11) tends to zero as $n \to \infty$. This completes the proof of Theorem 3. □

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Received by the editors: March 7, 2002.