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# SEQUENCES OF LINEAR OPERATORS RELATED TO CESÀRO - CONVERGENT SEQUENCES

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**Abstract.** Given a Cesàro-convergent sequence of real numbers $(a_n)_{n\in\mathbb{N}}$, a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of operators is defined on the Banach space $\mathcal{R}(I,F)$ of regular functions defined on $I=[0,1]$ and having values in a Banach space $F$,

$$\varphi_n(f)=\frac{1}{n}\sum_{k=1}^{n}a_k f\left(\tfrac{k}{n}\right).$$

It is proved that if, in addition, the sequence $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$ is bounded, then $\varphi_n(f)$ converges to $a\cdot\int_0^1 f$, where $a=\lim_{n\to\infty}\frac{a_1+\ldots+a_n}{n}$. The converse of this statement is also true. Another result is that the supplementary condition can be dropped if the operators are considered on the space $\mathcal{C}^1(I,F)$.

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## 1. INTRODUCTION

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. It will be called *Cesàro-convergent* if the sequence of its Cesàro (arithmetic) means is convergent, i.e.

$$\lim_{n\to\infty}\frac{a_1+\ldots+a_n}{n}\in\mathbb{R}.$$

For $x\in\mathbb{R}$, $\lfloor x\rfloor$ will denote the greatest integer number $n\le x$ (the integer part of $x$).

Given the interval $I=[0,1]$ and a Banach space $F\neq\{0\}$, we denote by $\mathcal{B}(I,F)$ the Banach space of bounded functions $f:I\to F$ endowed with the sup norm. The space $\mathcal{B}(I,F)$ contains as a subspace the set of "step-functions" $\mathcal{E}(I,F)=\{f:I\to F:\exists\, t_0,\ldots,t_n\in I,\ t_0=0<t_1<\ldots<t_n=1,\ \exists\, u_k\in F$ so that $f|_{(t_{k-1},t_k)}=u_k,\ k=1,\ldots,n\}$. In fact each $f\in\mathcal{E}(I,F)$ is a finite sum of functions having the form $\chi_{[\alpha,\beta]}\cdot u$, where $0\le\alpha\le\beta\le 1$, $u\in F$ and $\chi_{[\alpha,\beta]}$ is the characteristic function of the interval $[\alpha,\beta]$. We denote by $\mathcal{R}(I,F)$ the Banach space of *regular functions* (which admit side limits at each $t\in I$), endowed with the uniform norm $\|f\|=\sup_{t\in[0,1]}\|f(t)\|$. We mention that $\mathcal{R}(I,F)$ is the closure in $\mathcal{B}(I,F)$ of the subspace $\mathcal{E}(I,F)$, and it contains the

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Banach space of continuous functions $\mathcal{C}(I,F)$. More details on these spaces of functions are to be found in [3, p. 137].

We define a sequence of operators associated to $(a_n)_{n\in\mathbb{N}}$, namely $\varphi_n : \mathcal{R}(I,F) \to F,\ n \in \mathbb{N}$

$$\varphi_n(f) = \frac{1}{n}\sum_{k=1}^{n} a_k f\left(\frac{k}{n}\right). \tag{1}$$

Proposition 1. *The operator $\varphi_n$ is linear and continuous, and its norm is given by*

$$\|\varphi_n\| = \frac{1}{n}\sum_{k=1}^{n} |a_k|. \tag{2}$$

*Proof.* The linearity is straightforward. Because $\left\|f\left(\frac{k}{n}\right)\right\| \le \|f\|$, it follows

$$\|\varphi_n(f)\| \le \left(\frac{1}{n}\sum_{k=1}^{n} |a_k|\right)\cdot\|f\|, \tag{3}$$

hence $\varphi_n$ is also continuous. To obtain the norm of $\varphi_n$, we use the inequality (3) and the function

$$f_0(t) = \begin{cases} (\operatorname{sign} a_k)u, & \text{for } t = \frac{k}{n},\ k = 1,\dots,n \\ 0, & \text{otherwise,} \end{cases}$$

where $u \in F$ and $\|u\| = 1$. We have $f_0 \in \mathcal{E}(I,F) \subseteq \mathcal{R}(I,F)$, $\|f_0\| = 1$ and $\varphi_n(f_0) = \left(\frac{1}{n}\sum_{k=1}^{n}|a_k|\right)\cdot u$, hence the equality (2) follows. □

## 2. MAIN RESULTS

We are interested in finding conditions on the sequence $(a_n)_{n\in\mathbb{N}}$ in order to obtain the convergence of the sequence of linear operators (1). The theorem below guarantees the convergence of $(\varphi_n(f))_{n\in\mathbb{N}}$ for each regular function $f \in \mathcal{F}(I,F)$. Beside the condition of Cesàro-convergence for $(a_n)_{n\in\mathbb{N}}$, the boundedness of a certain sequence related to this is imposed.

Theorem 2. *Let there be given a regular function $f \in \mathcal{R}(I,F)$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers satisfying the conditions:*

1. *$(a_n)_{n\in\mathbb{N}}$ is Cesàro-convergent to $a$ ($\lim_{n\to\infty}\frac{a_1+\ldots+a_n}{n} = a$);*
2. *the sequence $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$ is bounded.*

*Then the sequence $(\varphi_n(f))_{n\in\mathbb{N}}$ is convergent and*

(4)
$$\lim_{n\to\infty} \varphi_n(f) = a \cdot \int_0^1 f.$$

*Proof.* At first we shall prove (4) for functions $f$ of the form

(5)
$$f = \chi_{[\alpha,\beta]} \cdot u, \text{ where } 0 \le \alpha \le \beta \le 1,\ u \in F.$$

We have

$$\varphi_n(f) = \Bigg(\frac{1}{n} \sum_{\substack{k\in\mathbb{N}\\ \alpha n \le k \le \beta n}} a_k\Bigg) \cdot u = \Bigg(\frac{1}{n} \sum_{\substack{k\in\mathbb{N}\\ k \le \beta n}} a_k\Bigg) \cdot u - \Bigg(\frac{1}{n} \sum_{\substack{k\in\mathbb{N}\\ k < \alpha n}} a_k\Bigg) \cdot u.$$

If $\alpha = 0$ the conclusion follows obviously.

For $\alpha > 0$ we denote $a_n^* = \frac{a_1+\ldots+a_n}{n}$ and we write the two sums in the above formula as

$$\sum_{\substack{k\in\mathbb{N}\\ k \le \beta n}} a_k = \lfloor \beta n \rfloor \cdot a^*_{\lfloor \beta n \rfloor}, \qquad \sum_{\substack{k\in\mathbb{N}\\ k < \alpha n}} a_k = \lfloor \alpha n \rfloor \cdot a^*_{\lfloor \alpha n \rfloor} - a_{\lfloor \alpha n \rfloor} \cdot \theta_n,$$

where

$$\theta_n = \begin{cases} 1, & \text{for } \alpha n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

We finally obtain

$$\varphi_n(f) = \Big(\frac{\lfloor \beta n \rfloor}{n} \cdot a^*_{\lfloor \beta n \rfloor} - \frac{\lfloor \alpha n \rfloor}{n} \cdot a^*_{\lfloor \alpha n \rfloor} + \frac{a_{\lfloor \alpha n \rfloor}}{n} \cdot \theta_n\Big) \cdot u.$$

We have $\lim\limits_{n\to\infty} a^*_{\lfloor \alpha n \rfloor} = a$; but $\frac{a_n}{n} = a_n^* - \Big(1 - \frac{1}{n}\Big) a^*_{n-1}$, hence $\lim\limits_{n\to\infty} \frac{a_n}{n} = 0$. It follows that in this case

$$\lim_{n\to\infty} \varphi_n(f) = (\beta a - \alpha a) \cdot u = a \cdot \int_0^1 f.$$

We consider now the general case $f \in \mathcal{R}(I, F)$. The sequence $\big(\frac{|a_1|+\ldots+|a_n|}{n}\big)_{n\in\mathbb{N}}$ being bounded, let us choose $M$ such that $\frac{|a_1|+\ldots+|a_n|}{n} \le M$ for each $n \in \mathbb{N}$; let also $\varepsilon > 0$ be an arbitrary constant. From the definition of the space $\mathcal{R}(I, F)$ it follows the existence of the functions $f_i$, $i = 1, \ldots, p$ of the type described in (5), with $\big\|f - \sum_{i=1}^p f_i\big\| < \varepsilon$. We have

$$\varphi_n(f) - a\int_0^1 f = \varphi_n(f - \sum_{i=1}^p f_i) + \sum_{i=1}^p \Big(\varphi_n(f_i) - a\int_0^1 f_i\Big) - a\int_0^1 \Big(f - \sum_{i=1}^p f_i\Big).$$

The norm of $\varphi_n$, as given by (2), is $\|\varphi_n\| = \frac{|a_1|+\ldots+|a_n|}{n}$, hence

(6)
$$\Big\|\varphi_n(f - \sum_{i=1}^p f_i)\Big\| \le \|\varphi_n\| \cdot \Big\|f - \sum_{i=1}^p f_i\Big\| \le M \cdot \varepsilon.$$

Taking into account the first part of the proof, for each $i = 1, \ldots, p$ there exists $n_i \in \mathbb{N}$ so that $\big\|\varphi_n(f_i) - a \cdot \int_0^1 f_i\big\| < \frac{\varepsilon}{p}$ for $n \ge n_i$. It follows that for

$n \geq \max\limits_{i=1,\ldots,p} n_i$ we have

$$\Big\| \sum_{i=1}^{p} (\varphi_n(f_i) - a \cdot \int_0^1 f_i) \Big\| \leq \varepsilon. \tag{7}$$

But

$$\Big\| a \int_0^1 (f - \sum_{i=1}^{p} f_i) \Big\| \leq |a| \cdot \varepsilon, \tag{8}$$

and the inequalities (6), (7) and (8) imply that

$$\Big\| \varphi_n(f) - a \cdot \int_0^1 f \Big\| \leq M \cdot \varepsilon + \varepsilon + |a| \cdot \varepsilon, \text{ for } n \geq \mathbb{N}.$$

It follows that the conclusion holds also for the general case $f \in \mathcal{F}(I, F)$. □

Remark 1. The Cesàro-convergence of $(a_n)_{n\in\mathbb{N}}$ in Theorem 2 does not necessarily imply the boundedness of $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$. For example, let the sequence be given by

$$a_n = \begin{cases} \sqrt{n}, & n \text{ odd} \\ -\sqrt{n-1}, & n \text{ even.} \end{cases}$$

Then

$$a_n^* = \begin{cases} 1/\sqrt{n}, & n \text{ odd} \\ 0, & n \text{ even,} \end{cases}$$

hence $\lim\limits_{n\to\infty} a_n^* = 0$, but $\lim\limits_{n\to\infty} \frac{|a_1|+\ldots+|a_n|}{n} = \lim\limits_{n\to\infty} |a_n| = \infty$. □

The condition of Cesàro-convergence imposed to the sequence $(a_n)_{n\in\mathbb{N}}$ in Theorem 2 is a natural one and cannot be relaxed, neither the boundedness of the sequence $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$. In fact, Theorem 2 does admit the following converse:

Theorem 3. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be the sequence* (1) *of linear operators associated to the sequence of real numbers $(a_n)_{n\in\mathbb{N}}$. If $\lim\limits_{n\to\infty} \varphi_n(f)$ exists for every $f \in \mathcal{C}(I,F) \subseteq \mathcal{R}(I,F)$, then:*

1. *$(a_n)_{n\in\mathbb{N}}$ is Cesàro-convergent to $a$ ($\lim\limits_{n\to\infty} \frac{a_1+\ldots+a_n}{n} = a$);*
2. *the sequence $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$ is bounded.*

*Proof.* The first conclusion follows by taking $f(t) = u$ for each $t \in I$, with $u \in F \setminus \{0\}$. In this case $\varphi_n(f) = \frac{a_1+\ldots+a_n}{n} u$.

The norm of the operators $\varphi_n$ in the space $\mathcal{C}(I,F)$ is the same as in (1). Indeed, in the proof of Proposition 1, the function $f_0$ can be modified to a continuous and piecewise affine one which takes also the values $(\operatorname{sign} a_k)u$ on the points $\frac{k}{n}$, $k = 1, \ldots, n$. From the principle of uniform boundedness [4, p. 66] the second conclusion follows. □

REMARK 2. Using a principle of condensation of singularities [2], one can prove that the convergence in (4) does not hold for "typical" continuous functions. Even stronger principles of condensation of singularities [1] may be applied. □

In what follows we shall prove that for the class of continuous functions having also a continuous derivative, the condition of boundedness of the sequence $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$ is no longer necessary. In this setting, the principle of uniform boundedness does not work, because $\mathcal{C}^1(I,F)$ endowed with the uniform norm is not a Banach space. The norm of $\varphi_n$ is still the same. In this case we have

THEOREM 4. *Let there be given a function $f \in \mathcal{C}^1(I,F)$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers which is Cesàro-convergent to $a$ ($\lim\limits_{n\to\infty} \frac{a_1+\ldots+a_n}{n} = a$). Then*

$$\lim_{n\to\infty} \varphi_n(f) = a \cdot \int_0^1 f. \tag{9}$$

*Proof.* We write $\varphi_n(f)$ successively as

$$\begin{aligned}
\varphi_n(f) &= \tfrac{1}{n} \sum_{k=1}^{n} \left(k a_k^* - (k-1)\, a_{k-1}^*\right) f\left(\tfrac{k}{n}\right) \\
&= \tfrac{1}{n} \sum_{k=1}^{n} k a_k^* f\left(\tfrac{k}{n}\right) - \tfrac{1}{n} \sum_{k=1}^{n-1} k a_k^* f\left(\tfrac{k+1}{n}\right) \\
&= \sum_{k=1}^{n-1} a_k^* \tfrac{k}{n} \left(f\left(\tfrac{k}{n}\right) - f\left(\tfrac{k+1}{n}\right)\right) + a_n^* f(1).
\end{aligned}$$

We bring now into the scene the continuous function $g$ given by $g(t) = t f'(t)$ and express $\varphi_n(f)$ in the form

$$\begin{aligned}
\varphi_n(f) &= -\sum_{k=1}^{n-1} a_k^* \tfrac{k}{n}\Big(f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right) - \tfrac{1}{n} f'(\tfrac{k}{n})\Big) - \tfrac{1}{n} \sum_{k=1}^{n-1} a_k^* \tfrac{k}{n} f'(\tfrac{k}{n}) + a_n^* f(1) \\
&= -\sum_{k=1}^{n-1} a_k^* \tfrac{k}{n}\Big(f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right) - \tfrac{1}{n} f'(\tfrac{k}{n})\Big) - \tfrac{1}{n} \sum_{k=1}^{n} a_k^* g(\tfrac{k}{n}) + \tfrac{1}{n} a_n^* f'(1) + a_n^* f(1).
\end{aligned} \tag{10}$$

Applying Theorem 2 for the function $g$ and for the sequence $(a_n^*)_{n\in\mathbb{N}}$ convergent to $a$, for which obviously $\lim\limits_{n\to\infty} \frac{a_1^*+\ldots+a_n^*}{n} = a$ and $\left(\frac{|a_1^*|+\ldots+|a_n^*|}{n}\right)_{n\in\mathbb{N}}$ is bounded (because of the convergence of $(a_n^*)_{n\in\mathbb{N}}$) we get

$$\lim_{n\to\infty} \tfrac{1}{n} \sum_{k=1}^{n} a_k^* g\left(\tfrac{k}{n}\right) = a \cdot \int_0^1 g = a \cdot f(1) - a \cdot \int_0^1 f$$

(the last equality is a consequence of an integration by parts). The function $f'$ being uniformly continuous on $I$, given $\varepsilon > 0$ and $n$ sufficiently large, we obtain as a consequence of a mean theorem [3, p. 154]

$$\left\|f\left(\tfrac{k+1}{n}\right)-f\left(\tfrac{k}{n}\right)-\tfrac{1}{n}f'\left(\tfrac{k}{n}\right)\right\|\leq\tfrac{1}{n}\sup_{t\in(\frac{k}{n},\frac{k+1}{n})}\left\|f'(t)-f'\left(\tfrac{k}{n}\right)\right\|<\tfrac{\varepsilon}{n},$$

hence

$$\left\|\sum_{k=1}^{n-1}a_k^*\tfrac{k}{n}\Big(f\left(\tfrac{k+1}{n}\right)-f\left(\tfrac{k}{n}\right)-\tfrac{1}{n}f'\left(\tfrac{k}{n}\right)\Big)\right\|\leq\sum_{k=1}^{n-1}\tfrac{M\varepsilon}{n^2}k=\tfrac{n-1}{2n}M\varepsilon\leq M\varepsilon,$$

where $M$ is a upper bound for the convergent sequence $(|a_n^*|)_{n\in\mathbb{N}}$. It follows that

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}a_k^*\tfrac{k}{n}\Big(f\left(\tfrac{k+1}{n}\right)-f\left(\tfrac{k}{n}\right)-\tfrac{1}{n}f'\left(\tfrac{k}{n}\right)\Big)=0.$$

We take the limit in (10) and get the conclusion. □

As an application of Theorem 2 we obtain a somehow surprising result, proved directly for differentiable functions with bounded derivative in [5]: For each $a\in[0,1]$, there exist $\varepsilon_n\in\{0,1\}$ such that

$$\lim_{n\to\infty}\tfrac{1}{n}\sum_{k=1}^{n}\varepsilon_k f\left(\tfrac{k}{n}\right)=a\cdot\int_0^1 f,\qquad\forall f\in\mathcal{R}(I,F).$$

To prove this equality, we choose $\varepsilon_n=a_n=\lfloor(n+1)a\rfloor-\lfloor na\rfloor$, $n\in\mathbb{N}$ which satisfy $\varepsilon_n\in\{0,1\}$ and $\lim_{n\to\infty}\frac{a_1+\ldots+a_n}{n}=a$.

**Open question**. It would be interesting to find out if the conclusion of Theorem 2 also holds for a class of functions more general than the regular ones as, for example, the Riemann integrable real-valued functions. For the class of Lebesgue integrable functions the result does not hold, as the function of Dirichlet type $f:I\to F=\mathbb{R}$,

$$f(t)=\begin{cases}\text{arbitrary}, & t\in[0,1]\cap\mathbb{Q}\\ 0, & \text{otherwise}\end{cases}$$

shows.

## REFERENCES

[1] Anisiu, V., *A principle of double condensation of singularities using σ-porosity*, "Babeş-Bolyai" Univ., Fac. of Math., Research Seminaries, Seminar on Math. Analysis, Preprint Nr. **7**, pp. 85–88, 1985.

[2] Cobzaş, S. and Muntean, I., *Condensation of singularities and divergence results in approximation theory*, J. Approx. Theory, **31**, pp. 148–153, 1981.

[3] Dieudonné, J., *Fondements de l'analyse moderne*, Paris, Gauthier-Villars, 1963.

[4] Dunford, N. and Schwartz, J. T., *Linear Operators. Part 1: General Theory*, John Wiley & Sons, New York, 1988.

[5] Trif, T., *On a problem from the Mathematical Contest, County Stage*, Gazeta Matematică CVI (11), pp. 394–396, 2001 (in Romanian).

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