SOME REMARKS ON THE MONOTONE ITERATIVE TECHNIQUE

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Abstract. We consider an abstract operator equation in coincidence form \( Lu = N(u) \) and establish some comparison results and existence results via the monotone iterative technique. We use a generalized iteration method developed by Carl-Heikkila (1999). An application to a boundary value problem for a second-order functional differential equation is considered.

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1. INTRODUCTION

Let \( X \) be a nonempty set and \( Z \) be an ordered metric space. Let us consider the operator equation of the form

\[
Lu = Nu,
\]

and the iterative scheme

\[
Lu_{n+1} = Nu_n,
\]

where \( L, N : X \to Z \).

In our work the operators \( L \) and \( N \) will satisfy some extended monotonicity conditions, which are described exactly in the following definition.

Definition 1.1. \( N \) is monotone increasing with respect to \( L \) if \( u_1, u_2 \in X \) and \( Lu_1 \leq Lu_2 \) imply that \( Nu_1 \leq Nu_2 \).

If in the last relation the reversed inequality holds, then \( N \) is monotone decreasing with respect to \( L \).

Let \( X \) be an ordered set. If \( Lu_1 \leq Lu_2 \) implies \( u_1 \leq u_2 \) then \( L \) is said to be inverse-monotone (see [8]) or of monotone-type (see [9]).

The plan of our paper is as follows. In Section 2 we deal with operator inequalities corresponding with [11] and extend the abstract Gronwall lemma of Rus [6]. Let us mention that the result from [6] generalize some results from [9] and [11]. In Section 3 we generalize some known existence results for equation (1.1) ([5, 10, 4, 1, 9]) involving monotone increasing or monotone decreasing operators. We shall use a generalized iteration method developed in [2]. In Section 4 we shall apply some of our results to implicit second...
order functional-differential equations. For another treatment of this type of functional-differential equations it can be seen [7].

2. OPERATOR INEQUALITIES IN ORDERED METRIC SPACES

In this section we extend the notion of Picard operator [6], in Definition 2.1, and the abstract Gronwall lemma of Rus [6], in Theorem 2.3. The above mentioned notion and result correspond, in our setting, with the case when $X = Z$ and $L$ is the identity mapping of $Z$. As a consequence of Theorem 2.3, we shall find a condition in Corollary 2.4, which assure the existence of ordered lower and upper solutions for equation (1.1).

**Definition 2.1.** $N$ is Picard with respect to $L$ if there exists a unique $v^* \in Z$ with the following properties.

(i) there exists $u^* \in X$ such that $Lu^* = Nu^* = v^*$;

(ii) $N(X) \subset L(X)$;

(iii) for every $u_0 \in X$ a sequence defined by (1.2) is such that $(Lu_n)_{n \geq 0}$ is convergent to $v^*$.

**Example 2.1.** If $X = Z$ and $N : Z \to Z$ is Picard [6] then $N$ is Picard with respect to $I$, the identity mapping of $Z$. □

**Example 2.2.** If $L$ is inversable and $N \circ L^{-1} : Z \to Z$ is Picard, then $N$ is Picard with respect to $L$. □

**Example 2.3.** Let $L, N : (0, \infty) \to (-1, \infty)$ be given by $L(u) = u^2 - 1$ and $N(u) = \sqrt{u}$. Then $N$ is Picard with respect to $L$. Let us mention that, also, $N$ is monotone increasing with respect to $L$. □

**Example 2.4.** If $Z$ is also a complete metric space, $L$ is surjective and $N$ is contraction with respect to $L$ then $N$ is Picard with respect to $L$. Let us mention that $N$ is contraction with respect to $L$ if there exists $0 < a < 1$ such that for all $u_1, u_2 \in X$, $d(Nu_1, Nu_2) \leq a \cdot d(Lu_1, Lu_2)$.

For the proof of this result, also known as the Coincidence Theorem of Goebel, we refer to [3]. □

**Lemma 2.2.** If $N$ is monotone increasing (or monotone decreasing or contraction) with respect to $L$ then

$L u_1 = L u_2$ implies $N u_1 = N u_2$.

If $L$ is inverse-monotone then $L$ is injective.

**Proof.** Let us consider only that $N$ is monotone increasing with respect to $L$. If $L u_1 = L u_2$ then $L u_1 \leq L u_2$ and $L u_2 \leq L u_1$. Thus, $N u_1 \leq N u_2$ and $N u_2 \leq N u_1$. This obviously implies the conclusion.

For the last statement we have to prove that, if $L$ is of monotone type, then $L u_1 = L u_2$ implies $u_1 = u_2$. This can be done like above. □
THEOREM 2.3. If $N$ is monotone increasing with respect to $L$ and $N$ is Picard with respect to $L$ then

(i) $L u_0 \leq N u_0$ implies $L u_0 \leq v^*$,

(ii) $L u_0 \geq N u_0$ implies $L u_0 \geq v^*$.

If, in addition, $L$ is of monotone type then

(j) $L u_0 \leq N u_0$ implies $u_0 \leq u^*$,

(jj) $L u_0 \geq N u_0$ implies $u_0 \geq u^*$.

Proof. Let us consider $u_0 \in X$ such that $L u_0 \leq N u_0$ and the sequence defined by (1.2) starting from $u_0$. The following relations hold,

$$L u_0 \leq N u_0 = L u_1 \leq N u_1 = L u_2 \leq N u_2 \leq \ldots .$$

Thus, for all $n \geq 0$,

$$L u_0 \leq N u_n,$$

and, passing to the limit when $n \to \infty$,

$$L u_0 \leq v^*.$$

The next relation can be proved similarly.

If, in addition, $L$ is inverse-monotone, then, by Lemma 2.2, $u^*$ given by Definition 2.1 is unique and, of course, $L u_0 \leq L u^*$ implies $u_0 \leq u^*$.

We say that $u^e \in X$ is a lower solution of (1.1) if

$$L u^e \leq N u^e ,$$

Similarly, $u^s \in X$ is an upper-solution of (1.1) if

$$L u^s \geq N u^s .$$

COROLLARY 2.4. Let us consider two operators $N, \tilde{N} : X \to Z$ such that they are monotone increasing with respect to $L$ and Picard with respect to $L$.

If

(2.1)

$$N u \leq u \leq \tilde{N} u , \quad \text{for all } u \in X,$$

then there exist $u$ a lower solution and $\tilde{u}$ an upper-solution of (1.1), such that

$$L u \leq L \tilde{u}.$$

If, in addition, $L$ is of monotone type then,

$$u \leq \tilde{u}.$$

Proof. $N$ and $\tilde{N}$ being Picard with respect to $L$, there exist $y$ such that

$$L y = N y ,$$

and $\tilde{u}$ such that

$$L \tilde{u} = \tilde{N} \tilde{u}.$$

Then, by (2.1) $L y \leq N y$ and $L \tilde{u} \geq N \tilde{u}$, which mean that $y$ is a lower solution and $\tilde{u}$ is a super-solution of (1.1).

Also by (2.1) the following inequality holds

$$L y \leq \tilde{N} y.$$
We apply now Theorem 2.3 for $\bar{N}$ and deduce that

$$Lu \leq L\bar{u}.$$  

The last part of the conclusion follows in an obvious way. □

3. OPERATOR EQUATIONS IN ORDERED BANACH SPACES

In this section we shall establish two existence results for equation (1.1), involving an operator $N$ which is increasing with respect to $L$, in Theorem 3.2, respectively monotone decreasing with respect to $L$, in Theorem 3.3 We shall use a generalized iteration method developed in [2]. As it is mentioned in [2], this method enlarges the range of applications since neither $L$ nor $N$ need be continuous. In this spirit, Theorem 3.2 generalizes Theorem 3.1 in [4], and Theorem 3.3 generalizes Theorem 3 in [10] and Theorem 2 in [5] (these are given in the case $X = Z$ and $L = I$).

The following result is Proposition 3.4 from [2] and we shall use it to derive Theorem 3.2.

**Proposition 3.1.** Assume that the following conditions hold.

(i) There exists $u_\bar{u}$ a lower solution of (1.1), $u_\bar{u} \in W \subset X$;

(ii) $N$ is monotone increasing with respect to $L$;

(iii) $L(W)$ is an ordered metric space and if $(u_n)$ is a sequence in $W$ such that the sequences $(Lu_n)$ and $(Nu_n)$ are increasing, then $(Nu_n)$ converges in $L(W)$.

Then (1.1) has a solution $u_*$ with the property

$$Lu_* = \min \{Lw \in L(W) \mid Lu \leq Lw \text{ and } Lw \geq Nw \}.$$  

If, in addition, $W$ is an ordered space and $L$ is of monotone type, then $u_*$ is the minimal solution of (1.1) in $W_0 = \{u \in W \mid Lu \leq Lu\}$.

We notice that the dual result is valid.

In the following results, i.e. Theorem 3.2 and Theorem 3.3 $Z$ will be an ordered Banach space (OBS) with a normal cone $K$.

Let us remember, (see [4, 11, 10]) that the cone $K = \{v \in Z \mid v \geq 0\}$ is said to be normal if there exists $\delta > 0$ such that $0 \leq v \leq w$ implies $|v| \leq \delta |w|$.

For $v \leq w$ the order interval $[v, w]$ is the set of all $u \in Z$ such that $v \leq u \leq w$. Every order interval for an OBS is bounded if and only if the cone $K$ is normal.

In an OBS with a normal cone, every monotone increasing sequence which has a convergent subsequence, is convergent.

A cone $K$ is said to be regular if every monotone increasing sequence contained in some order interval, is convergent.

**Theorem 3.2.** Assume that the following conditions hold.

(i) $u$ is a lower solution and $\bar{u}$ is a super-solution of (1.1) with $Lu \leq L\bar{u}$;

(ii) $N$ is monotone increasing with respect to $L$;

(iii) $[Lu, L\bar{u}] \subset L(X)$;
By Lemma 2.2, \( \tilde{N} \) does not depend on the choice of \( \tilde{u} \), thus the operator \( \tilde{N} : X \to Z \) is well-defined.

(iv) \( K \) is regular or \([N\tilde{u}, N\tilde{u}] \cap N(X)\) is a compact subset of \( Z \).

Then \( (1.1) \) has a solution \( u_* \) with the property

\[ Lu_* = \min \{ Lw \in [L\tilde{u}, L\tilde{u}] \mid Lw \geq Nu \} \]

and a solution \( u^* \) with the property

\[ Lu^* = \max \{ Lw \in [L\tilde{u}, L\tilde{u}] \mid Lw \leq Nu \}. \]

If, in addition, \( L \) is of monotone type, then \( u_* \) is the minimal solution, and \( u^* \) the maximal solution of \( (1.1) \) in \([y, \bar{u}]\).

**Proof.** Let us consider \( W = \{ u \in X \mid Ly \leq Lu \leq L\tilde{u} \} \). Then, using also (iii), \( L(W) = [L\tilde{u}, L\tilde{u}] \), which is a closed subset of \( Z \), thus is an ordered metric space.

Let \( (u_n) \) be a sequence in \( W \) such that \( (Lu_n) \) and \( (Nu_n) \) are increasing. Using (i) and (ii), \( Ly \leq Lu_n \leq L\tilde{u} \) imply that \( Ly \leq Nu_n \leq Nu \leq N\tilde{u} \leq L\tilde{u} \). Then \( (Nu_n) \) is an increasing sequence in the bounded (because \( K \) is normal) interval \( L(W) \).

If \( K \) is regular, then \( (Nu_n) \) converges.

If \([N\tilde{u}, N\tilde{u}] \cap N(X)\) is compact, then \( (Nu_n) \) has a convergent subsequence. By the monotonicity of the sequence \( (Nu_n) \), it converges.

All the hypotheses of Proposition 3.1 are fulfilled. Hence, the conclusion follows.

**Remark.** If, in addition to the hypotheses of Theorem 3.2, \( N \) is continuous with respect to \( L \) then \( u^* \) can be obtained by \( (1.2) \) starting from \( u_i \), in the sense that a sequence defined by \( (1.2) \) with \( u_0 = y \) is such that \( (Lu_n) \) converges to \( Lu^* \).

Let us mention that \( N \) is said to be continuous with respect to \( L \) if for every sequence \( (Lu_n) \) from \( L(X) \) convergent to \( Lu^* \in L(X) \), the sequence \( (Nu_n) \) converges to \( Nu^* \).

**Theorem 3.3.** Assume that the following conditions hold.

(i) \( Lu \geq 0 \) implies \( Nu \geq 0 \);

(ii) \( N \) is monotone decreasing with respect to \( L \);

(iii) if \( u_0 \) and \( u_1 \) are such that \( Lu_0 = 0 \), \( Nu_0 = Lu_1 \) then \( Nu_1 > 0 \) and \([0, Lu_1] \subset L(X)\);

(iv) there exists \( \alpha \in (-1, 0) \) such that \( Nu \mu \leq \mu^\alpha Nu \) for all \( u \in X \) with \( 0 \leq Lu \leq Lu_1 \), for \( u_\mu \) given by \( Lu_\mu = \mu Lu \), and for all \( \mu \in (0, 1) \);

(v) for every \( v, w \) with \( 0 < v \leq w \leq Lu_1 \) there is \( \mu \in (0, 1) \) such that \( \mu v \leq w \);

(vi) the cone \( K \) is regular or \([Nu_1, Nu_0] \cap N(X)\) is a compact subset of \( Z \).

Then \( (1.1) \) has a solution, \( u^* \) with \( Lu^* > 0 \).

**Proof.** For every \( u \in X \), if \( \tilde{u} \) is such that \( Nu = L\tilde{u} \), let us define \( \tilde{N}u = Nu \).

By Lemma 2.2, \( \tilde{N}u \) does not depend on the choice of \( \tilde{u} \), thus the operator \( \tilde{N} : X \to Z \) is well-defined.
\( \tilde{N} \) is monotone increasing with respect to \( L \). Indeed, \( Lu_1 \leq Lu_2 \Rightarrow Lu_1 = Nu_1 \geq Nu_2 = L\tilde{u}_2 \Rightarrow \tilde{N}u_1 \leq \tilde{N}\tilde{u}_2 = Nu_2 \).

Let us consider also \( u_2, u_3 \) such that \( Nu_1 = Lu_2 \) and \( Nu_2 = Lu_3 \). By (i) and (iii), \( Lu_1 \geq 0 = Lu_0 \), which implies, by (ii), that \( Nu_1 \leq Nu_0 \). Using the definitions of \( u_2 \) and \( u_1 \), the following relation holds.

\[
\text{(3.1)} \quad Lu_2 \leq Lu_1.
\]

We shall focus our attention to the equation \( \text{(3.2)} \quad Lu = \tilde{N}u \).

We shall prove that \( u_2 \) is a lower solution and \( u_1 \) is an upper solution of \( \text{(3.2)} \).

This follows by the following implications.

\[
\text{If } Nu_1 \geq 0 = Lu_0 \Rightarrow Lu_2 \geq Lu_0 \Rightarrow Nu_2 \leq Nu_0 \Rightarrow \tilde{N}u_1 \leq \tilde{N}u_2 \nabla.
\]

We use Theorem 3.2 and deduce that equation \( \text{(3.2)} \) has a solution \( u^* \), i.e.

\[
\text{(3.1)} \quad Lu^* = \tilde{N}u^*,
\]

with the property

\[
Lu^* = \min \{Lw \in [Lu_2, Lu_1] \mid Lw \geq Nu \}.
\]

Let us consider \( \tilde{u}^* \) such that

\[
\text{If } Lu^* = L\tilde{u}^* \text{ then, the existence of a solution for } \text{(1.1)} \text{ is proved. Using (iii), (iv) and (v) we shall prove that this always holds. First, let us notice that } 0 < L\tilde{u}^* \leq L\tilde{u}^* \leq Lu_1 \text{. According to (v), let } \mu_0 = \sup \{\mu \in (0, 1) \mid \mu L\tilde{u}^* \leq Lu^* \}. \quad \text{Clearly, } \mu_0 L\tilde{u}^* \leq Lu^* \text{. We have to prove that } \mu_0 = 1. \text{ Then, } L\tilde{u}^* = Nu^* \leq N\tilde{u}_{\mu_0}^* = \mu_0^\alpha \tilde{N}u^* = \mu_0^\alpha Lu^* \text{. Here, } \tilde{u}_{\mu_0}^* \text{ is such that } L\tilde{u}_{\mu_0}^* = \mu_0 L\tilde{u}^* \text{. Consequently, } \mu_0^\alpha \leq \mu_0, \text{ that is } -\alpha \geq 1, \text{ a contradiction. Thus, } Lu^* = L\tilde{u}^*. \]

\[ \Box \]

4. APPLICATION

In this section we shall establish a weak maximum principle for the functional-differential operator

\[
Lu = -u'' - \lambda u(g(x))
\]

and an existence result for the following boundary value problem for a second order implicit functional-differential equation.

\[
\text{(4.1)} \quad \begin{cases}
-u''(x) = f(x, u(g(x)), u(x), -u''(x), u(x)), \text{ a.a. } x \in (0, 1) \\
u \in H^2(0, 1) \cap H^1_0(0, 1).
\end{cases}
\]
Let us list the following hypotheses.

\((g1)\) the function \(g : [0, 1] \rightarrow [0, 1]\) is continuous.

\((f1)\) the function \(f : (0, 1) \times \mathbb{R}^3 \rightarrow \mathbb{R}\) is Caratheodory and there exists a continuous function \(\varphi : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}\) such that

\[|f(x, u, v, w)| \leq \varphi(x, u, v), \text{ a.a. } x \in (0, 1), u, v, w \in \mathbb{R}.\]

\((f2)\) \(f\) is monotone increasing with respect to the last three variables.

Let us denote

\[Nu = f(x, u(x), u(g(x)), -u''),\]

\[Z = L^2(0, 1), \quad X = H^2(0, 1) \cap H^1_0(0, 1).\]

Then, we obtain two operators \(L, N : X \rightarrow Z\) and the BVP can be written in the following form (with \(\lambda = 0\)).

\[(4.2) \quad Lu = Nu, \quad u \in X.\]

Let us notice that \((f1)\) and the inclusion \(X \subset C[0, 1]\) imply that \(N\) is well-defined. Also, for our existence result, we shall not need another growth condition for the function \(f\).

Next we shall prove that, when \(0 \leq \lambda < 8\), the weak maximum principle holds for the functional-differential operator \(L\).

**Theorem 4.1.** If \(0 \leq \lambda < 8\) then \(L : X \rightarrow Z\) is surjective and it is inverse-monotone.

**Proof.** In order to prove that \(L\) is surjective we study the solvability of the following equation for an arbitrary \(w \in Z\).

\[(4.3) \quad Lu = w, \quad u \in X.\]

Let us consider the following integral operator.

\[A_w : C[0, 1] \rightarrow C[0, 1], \quad A_w u = \int_0^1 G(x, s)[\lambda u(g(s)) + w(s)]ds.\]

The Green function \(G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}\) is given by
\[ G(x, s) = \begin{cases} 
    s(1 - x), & \text{if } s \leq x \\
    x(1 - s), & \text{if } s \geq x.
\end{cases} \]

Then equation (4.3) is equivalent to
\[ A_w u = u, \quad u \in C[0, 1]. \]

By a straightforward calculation, the following relation can be proved
\[ \|A_w u_1 - A_w u_2\|_C \leq \lambda \cdot \frac{1}{5}\|u_1 - u_2\|_C. \]

Thus, \( A_w \) is a contraction on the Banach space \( C[0, 1] \), so it has a unique fixed point. Hence, \( L \) is surjective.

In order to prove that \( L \) is inverse-monotone, because \( L \) is linear it is sufficient to prove that \( Lu \leq 0 \) implies \( u \leq 0 \).

Let \( u^* \in X \) be such that \( Lu^* \leq 0 \). Let us denote by \( w^*(x) = Lu^*(x) \). Then \( w^*(x) \leq 0 \) and \( A_w u^* = u^* \).

The operator \( A_{w^*} \) is Picard and monotone increasing and, in this case, it is easy to see that \( A(0) \leq 0 \). Then, by Theorem 2.3 (or Theorem 4.1 in [6]) \( u^* \leq 0 \). □

The following theorem is an existence result for the BVP considered at the beginning of this section.

**Theorem 4.2.** If conditions (g1), (f1) and (f2) hold and there exists a sub-solution \( \overline{u} \) and an upper solution \( \bar{u} \) for problem (6) with
\[ -\overline{u}'' \leq -\bar{u}'' \]
then (6) has a solution.

**Proof.** This follows easily by Theorem 3.2. Let us omit the details and notice only some useful facts.

\( Z = L^2(0, 1) \) is an ordered Banach space with a regular cone (see [1]).

\[ [Lu, \bar{u}] \subset L(X) \] because \( L \) is surjective.

The condition (f2) and that \( L \) is inverse-monotone imply that \( N \) is monotone increasing with respect to \( L \). □

**REFERENCES**


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