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L^p-APPROXIMATION $(p \geq 1)$ BY STANCU-KANTOROVICH POLYNOMIALS

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Abstract. We establish direct and converse estimates for a generalized Kantorovich polynomial operator depending on a positive parameter.

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1. INTRODUCTION

In [6] is defined the following Kantorovich type polynomial:

$$K_n^\alpha(f,x) = (n+1)\sum_{k=0}^{n} w_{n,k}^\alpha(x) \cdot \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\,\mathrm{d}u, \tag{1}$$

where $f \in L^1[0,1]$, $\alpha \geq 0$, $x \in [0,1]$ and

$$\begin{aligned} w_{n,k}^\alpha(x) &= \tbinom{n}{k}\frac{x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}}, \\ x^{(k,-\alpha)} &= x(x+\alpha)\dots(x+(k-1)\alpha). \end{aligned} \tag{2}$$

For $\alpha = 0$, K_n^0 is the Kantorovich operator K_n given by

$$K_n(f,x) = (n+1)\sum_{k=0}^{n} \tbinom{n}{k} x^k(1-x)^{n-k}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\mathrm{d}u. \tag{3}$$

The positive linear polynomial operator K_n^α is the Kantorovich variant of the following Stancu operator introduced in [7]:

$$B_n^\alpha(f,x) = \sum_{k=0}^{n} w_{n,k}^\alpha(x)\, f\left(\tfrac{k}{n}\right),$$

where f is a real function on $[0,1]$, $x \in [0,1]$ and $\alpha \geq 0$. For $f \in C[0,1]$ it can be found in [2] an asymptotic relation of Voronovskaja type and direct results for the operator K_n^α, respectively.

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The aim of the present paper is to give direct and converse results for the operators (1)–(2) in the spaces $L^p[0,1]$, $1 \le p \le \infty$ endowed with the norm

$$\|f\|_p = \Big(\int_0^1 |f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}}, \;\; 1 \le p < \infty$$

(for $p = \infty$ we will always consider $C[0,1]$ instead of $L^\infty[0,1]$ with $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$).

In the following $\omega(f,\delta)_p$, $\omega^2(f,\delta)_p$ and $\omega_\varphi^2(f,\delta)_p$ denote the usual modulus of continuity and the second modulus of smoothness of $f \in L^p[0,1]$, $1 \le p \le \infty$ and the second modulus of smoothness of Ditzian-Totik [3] defined by $\omega_\varphi^2(f,\delta)_p = \sup_{0<h\le\delta} \|\Delta^2_{h\varphi(x)} f(x)\|_p$, where $f \in L^p[0,1]$, $1 \le p \le \infty$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. Furthermore, we shall use the following K-functionals:

$$(4)\;\; K_{2,\varphi}(f,t^2)_p = \inf\{\|f-g\|_p + t^2\|\varphi^2 g''\|_p \;:\; g \in C^2[0,1]\},$$

$$(5)\;\; \bar{K}_{2,\varphi}(f,t^2)_p = \inf\{\|f-g\|_p + t^2\|\varphi^2 g''\|_p + t^4\|g''\|_p \;:\; g \in C^2[0,1]\}$$

(see [3]) and

$$(6)\qquad K(f,t)_p = \inf\{\|f-g\|_p \;+\; t^2\; \|(\varphi^2 g')'\|_p \;:\; g \in C^2[0,1]\}$$

in view of [5].

Before we state our theorems let us observe the dependence of the parameter α of n such that $\alpha = \alpha(n) \to 0$, as $n \to \infty$, when we approximate f by $K_n^\alpha f$ in $L^p[0,1]$, $1 \le p \le \infty$. Indeed, in this case $\lim_{n\to\infty} \|K_n^\alpha f - f\|_p = 0$ and in particular we have $\lim_{n\to\infty} \|K_n^\alpha(t^2,x) - x^2\|_1 = 0$. Simple computations show (or see [2, Lm 1.1, p. 72]) that

$$K_n^\alpha((t-x)^2,x) \;=\; x(1-x) \cdot \tfrac{\alpha(n^2-1)+(n-1)}{(\alpha+1)(n+1)^2} + \tfrac{1}{3(n+1)^2}$$

and

$$\begin{aligned} K_n^\alpha((t-x)^2,x) &= K_n^\alpha(t^2,x) - x^2 - 2\,x\,K_n^\alpha(t-x,x) \\ &= K_n^\alpha(t^2,x) - x^2 - \tfrac{x(1-2x)}{n+1}. \end{aligned}$$

Hence

$$\|K_n^\alpha((t-x)^2,x)\|_1 = \tfrac{1}{6} \cdot \tfrac{\alpha(n^2-1)+(n-1)}{(\alpha+1)(n+1)^2} + \tfrac{1}{3(n+1)^2}$$

and

$$\|K_n^\alpha((t-x)^2,x)\|_1 \le \|K_n^\alpha(t^2,x) - x^2\|_1 + \tfrac{1}{4(n+1)} \to 0, \qquad \text{as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} \tfrac{\alpha(n^2-1)+(n-1)}{(\alpha+1)(n+1)^2} = 0$$

which is possible when $\alpha = \alpha(n) \to 0$, as $n \to \infty$.

In the following the constant C denotes a positive constant which can be different at each occurrence.

2. DIRECT THEOREMS

Our main result is the following

THEOREM 1. *Let* $f \in L^p[0,1]$ *where* $1 \leq p \leq \infty$. *Then there exists* $C > 0$ *independent of* n *and* f *such that*

(i) $\|K_n^\alpha f - f\|_p \leq C \cdot (\omega_\varphi^2(f, n^{-1/2})_p + n^{-1}\|f\|_p)$, *for* $\alpha = \alpha(n) = \mathcal{O}(n^{-1})$ *and* $1 < p \leq \infty$;

(ii) $\|K_n^\alpha f - f\|_1 \leq C \cdot (\omega_\varphi^2(f, n^{-1/2})_1 + n^{-1}\|f\|_1)$, *for* $\alpha = \alpha(n) = \mathcal{O}(n^{-4})$.

Proof of Theorem 1. (i) It is known the following representation of K_n^α by means of K_n [6]:

$$K_n^\alpha(f,x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot K_n(f,t)\mathrm{d}t,$$

where $B(a,b)$ denotes the Beta function. By expanding $K_n f$ by Taylor's formula we get

$$K_n(f,t) = K_n(f,x) + K_n'(f,x)(t-x) + \int_x^t (t-u)K_n''(f,u)\mathrm{d}u.$$

Hence

$$\begin{aligned}(7)\qquad & K_n^\alpha(f,x) - K_n(f,x) = \\ &= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1}\,[\,K_n(f,t) - K_n(f,x)\,]\mathrm{d}t \\ &= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\Big[K_n'(f,x)(t-x) + \\ &\qquad\qquad + \int_x^t (t-u)K_n''(f,u)\mathrm{d}u\Big]\mathrm{d}t.\end{aligned}$$

By simple computations we obtain

$$(8)\qquad \int_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1}(t-x)\mathrm{d}t = 0$$

and

$$(9)\qquad \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}(t-x)^2\mathrm{d}t = \frac{\alpha x(1-x)}{1+\alpha} \cdot B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right).$$

Then, in view of (7), (8) and [3, Lm. 9.6.1, p. 140] we have

$$\begin{aligned}(10)\qquad & |K_n^\alpha(f,x) - K_n(f,x)| \leq \\ &\leq \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot \frac{|t-x|}{x(1-x)}\left|\int_x^t u(1-u)|K_n''(f,u)|\mathrm{d}u\right|\mathrm{d}t.\end{aligned}$$

In what follows the proof is standard using the Hardy-Littlewood maximal function:

$$M(G,x) = \sup_t \left| \frac{1}{x-t}\int_x^t G(v)\mathrm{d}v \right|,$$

where $G(v)=v(1-v)|K_n''(f,v)|$, $v\in[0,1]$. Then, by (10), (9) and the inequality $\|M(G)\|_p \le p/(p-1)\ \|G\|_p$, true for $1<p\le\infty$, we obtain

$$
\begin{aligned}
(11)\qquad &\|K_n^\alpha(f,x)-K_n(f,x)\|_p \le \\
&\le \Big\| \tfrac{1}{B(\frac{x}{\alpha},\frac{1-x}{\alpha})}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}\ (1-t)^{\frac{1-x}{\alpha}-1}\cdot\tfrac{(t-x)^2}{x(1-x)}\cdot M(G,x)\mathrm{d}t\ \Big\|_p \\
&\le \Big\| \tfrac{1}{B(\frac{x}{\alpha},\frac{1-x}{\alpha})}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}\ (1-t)^{\frac{1-x}{\alpha}-1}\cdot\tfrac{(t-x)^2}{x(1-x)}\ \mathrm{d}t\ \Big\|_{C[0,1]}\cdot\|M(G,x)\|_p \\
&\le C\cdot\tfrac{\alpha}{1+\alpha}\|\varphi^2K_n''(f)\|_p.
\end{aligned}
$$

On the other hand, in view of [3, (9.3.5) and (9.3.7), p. 118] we have from (11) for every $g\in C^2[0,1]$:

$$
\begin{aligned}
\|K_n^\alpha f-K_nf\|_p &\le C\ \tfrac{\alpha}{1+\alpha}\big(\|\varphi^2K_n''(f-g)\|_p+\|\varphi^2K_n''(g)\|_p\big)\\
&\le C\ \tfrac{\alpha}{1+\alpha}\big(n\ \|f-g\|_p+\|\varphi^2g''\|_p\big)\\
&= C\ \tfrac{\alpha n}{1+\alpha}\big(\|f-g\|_p+\tfrac{1}{n}\|\varphi^2g''\|_p\big).
\end{aligned}
$$

Hence $\|K_n^\alpha f-K_nf\|_p\le CK_{2,\varphi}(f,n^{-1})_p$, for $\alpha=\mathcal{O}(n^{-1})$. Using the equivalence of $K_{2,\varphi}(f,n^{-1})_p$, with $\omega_\varphi^2(f,n^{-1/2})_p$ (see [3, Th. 2.1.1, p. 11]) and [3, Th. 9.3.2, p. 117], we get (i).

(ii) In this case we need the following lemma:

Lemma 2. *If $\alpha=\alpha(n)=\mathcal{O}(n^{-4})$ then for $w_{n,k}^\alpha$ defined by* (2) *we have $\|w_{n,k}^\alpha\|_1=\mathcal{O}(1/(n+1))$. Moreover, for $K_n^\alpha f$ defined by* (1) *there exists C independent of n and f such that $\|K_n^\alpha f\|_1\le C\|f\|_1$.*

Proof of Lemma 2. Let $x\in[1/(2n^2),1-(1/(2n^2))]$. Then

$$
\begin{aligned}
(12)\qquad &w_{n,k}^\alpha(x)/w_{n,k}^0(x)= \\
&=\tfrac{1}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)}\cdot\big(1+\tfrac{\alpha}{x}\big)\Big(1+\tfrac{2\alpha}{x}\Big)\cdot\ldots\cdot\Big(1+\tfrac{(k-1)\alpha}{x}\Big)\cdot\\
&\quad\cdot\Big(1+\tfrac{\alpha}{1-x}\Big)\Big(1+\tfrac{2\alpha}{1-x}\Big)\dots\Big(1+\tfrac{(n-k-1)\alpha}{1-x}\Big)\\
&\le\big(1+\tfrac{n\alpha}{x}\big)^{k-1}\ \Big(1+\tfrac{n\alpha}{1-x}\Big)^{n-k-1}\\
&\le(1+2\ \alpha\ n^3)^{k-1}\cdot(1+2\ \alpha\ n^3)^{n-k-1}=(1+2\ \alpha\ n^3)^{n-2},
\end{aligned}
$$

where $0\le k\le n$. Since $w_{n,k}^\alpha$ is a polynomial of degree n we can use [3, Th. 8.4.8, p. 108] translated from $[-1,1]$ to $[0,1]$ to obtain the estimate

$$
\begin{aligned}
\|w_{n,k}^\alpha\|_1 &\le C\,\|w_{n,k}^\alpha\|_{L^1[1/(2n^2),1-(1/(2n^2))]}\\
&\le C\,(1+2\ \alpha\ n^3)^{n-2}\ \|w_{n,k}^0\|_{L^1[1/(2n^2),1-(1/(2n^2))]}\\
&\le C\,(1+2\ \alpha\ n^3)^{n-2}\|w_{n,k}^0\|_1.
\end{aligned}
$$

On the other hand $\|w_{n,k}^0\|_1=\mathcal{O}(1/(n+1))$ and

$$(1+2\,\alpha\,n^3)^{n-2}=\big[(1+2\,\alpha\,n^3)^{1/(2\alpha n^3)}\big]^{2\alpha n^3(n-2)}=\mathcal{O}(1)$$

for $\alpha = \mathcal{O}(n^{-4})$. Hence $\|w_{n,k}^{\alpha}\|_1 = \mathcal{O}(1/(n+1))$.

Using this last estimation we obtain

$$\begin{aligned}\int_0^1 |K_n^{\alpha}(f,x)|\mathrm{d}x &\leq \int_0^1 \sum_{k=0}^{n} w_{n,k}^{\alpha}(x)\Big\{(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(u)|\mathrm{d}u\Big\}\mathrm{d}x \\ &\leq \sum_{k=0}^{n}\int_0^1 w_{n,k}^{\alpha}(x)\mathrm{d}x(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(u)|\mathrm{d}u \\ &\leq C\|f\|_1,\end{aligned}$$

which was to be proved. □

Now let $E(n) = [1/n, 1-(1/n)]$ and $x \in E(n)$. By expanding the function $g \in C^2[0,1]$ by Taylor's formula we get

$$g(t) = g(x) + g'(x)\ (t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u.$$

Hence, in view of $K_n^{\alpha}(t-x,x) = (1-2x)/(2(n+1))$ [2, Lm. 1.1, p. 72] we have

$$(13)\qquad \|K_n^{\alpha}g - g\|_{L^1(E(n))} \leq$$

$$\begin{aligned}&\leq \int_{E(n)} |g'(x)|\cdot|K_n^{\alpha}(t-x,x)|\mathrm{d}x + \int_{E(n)}\Big|\,K_n^{\alpha}\Big(\int_x^t (t-u)g''(u)\mathrm{d}u;x\Big)\,\Big|\,\mathrm{d}x \\ &= \int_{E(n)} |g'(x)|\cdot\Big|\,\tfrac{1-2x}{2(n+1)}\,\Big|\,\mathrm{d}x + \int_{E(n)}\Big|\,K_n^{\alpha}\Big(\int_x^t (t-u)g''(u)\mathrm{d}u;x\Big)\,\Big|\,\mathrm{d}x \\ &\leq \tfrac{1}{2(n+1)}\cdot\|g'\|_1 + \int_{E(n)}\Big|\,K_n^{\alpha}\Big(\int_x^t (t-u)g''(u)\mathrm{d}u;x\Big)\,\Big|\,\mathrm{d}x.\end{aligned}$$

Furthermore we shall use some typical ideas which can be found in [3, Ch. 9, p. 145]. Thus we point out only that estimate which is different of the standard one. This happens at the evaluation of I_n, more precisely we have to prove for $x \in E(n)$

$$\sum_{k\in D(l,n,x)} w_{n,k}^{\alpha}(x)\big(|\tfrac{k}{n}-x| + \tfrac{1}{n}\big) \leq \tfrac{C}{(l+1)^4}\cdot\big(\varphi(x)n^{-\frac{1}{2}}\big)$$

(see [3, (9.6.11), p. 146]). But this follows from $w_{n,k}^{\alpha}(x) \leq w_{n,k}^{0}(x)\cdot(1+n^2\alpha)^{n-2}$ (see the proof of Lemma 2), where $x \in E(n)$ and $\alpha = \mathcal{O}(n^{-4})$ implies $(1+n^2\ \alpha)^{n-2} = \mathcal{O}(1)$, as $n \to \infty$. Therefore

$$\begin{aligned}\sum_{k\in D(l,n,x)} w_{n,k}^{\alpha}(x)\left(|\tfrac{k}{n}-x| + \tfrac{1}{n}\right) &\leq \sum_{k\in D(l,n,x)} C\ w_{n,k}^{0}(x)\left(\big|\ \tfrac{k}{n}-x\ \big| + \tfrac{1}{n}\right) \\ &\leq \tfrac{C}{(l+1)^4}\cdot\big(\varphi(x)\ n^{-1/2}\big)\end{aligned}$$

using [3, (9.6.11), p. 146]. In conclusion

$$\int_{E(n)}\Big|\,K_n^{\alpha}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\Big|\,\mathrm{d}x \leq \tfrac{C}{n}\|\varphi^2 g''\|_1$$

and from (13) we obtain

$$\|K_n^\alpha g-g\|_{L^1(E(n))}\le \tfrac{C}{n}\left(\|g'\|_1+\|\varphi^2g''\|_1\right).$$

Hence, in view of [3, §9.6, p. 140] we get

$$\|K_n^\alpha g-K_ng\|_{L^1(E(n))}\le \tfrac{C}{n}\left(\|g'\|_1+\|\varphi^2g''\|_1\right). \tag{14}$$

Now let $x\in[0,1]\setminus E(n)$ and $g\in C^2[0,1]$. Using the modified Taylor's formula

$$\begin{aligned}K_n(g,t)=&K_n(g,x)+K_n'(g,x)(t-x)+\\&+(t-x)\int_0^1\left[K_n'(g,x+u(t-x))-K_n'(g,x)\right]\mathrm{d}u,\end{aligned}$$

(8) and the representation of K_n^α by means of K_n we obtain

$$\begin{aligned}&\mid K_n^\alpha(g,x)-K_n(g,x)\mid=\\&=\Big|\tfrac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\Big[K_n'(g,x)(t-x)+\\&\quad+(t-x)\int_0^1\{K_n'(g,x+u(t-x))-K_n'(g,x)\}\mathrm{d}u\Big]\mathrm{d}t\,\Big|\\&\le\tfrac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}|t-x|\cdot\\&\quad\cdot\int_0^1\mid K_n'(g,x+u(t-x))-K_n'(g,x)\mid\mathrm{d}u\mathrm{d}t\le\end{aligned}\tag{15}$$

$$\begin{aligned}&\le\tfrac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}|t-x|\cdot\omega\big(K_n'(g),|t-x|\big)_1\mathrm{d}t\\&\le\tfrac{1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}|t-x|\big(n^2|t-x|+1\big)\omega\big(K_n'(g),n^{-2}\big)_1\mathrm{d}t\\&=\tfrac{\omega(K_n'(g),n^{-2})_1}{B\left(\frac{x}{\alpha},\frac{1-x}{\alpha}\right)}\cdot\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\left[n^2(t-x)^2+|t-x|\right]\mathrm{d}t.\end{aligned}$$

On the other hand, in view of Cauchy's inequality and (9) we have

$$\begin{aligned}&\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}|t-x|\ \mathrm{d}t\le\\&\le\left(\int_0^1 t^{\frac{x}{\alpha}-1}\ (1-t)^{\frac{1-x}{\alpha}-1}\mathrm{d}t\right)^{\frac12}\cdot\left(\int_0^1 t^{\frac{x}{\alpha}-1}\ (1-t)^{\frac{1-x}{\alpha}-1}\ (t-x)^2\mathrm{d}t\right)^{\frac12}\\&=\sqrt{\tfrac{\alpha x(1-x)}{1+\alpha}}\cdot B\left(\tfrac{x}{\alpha},\tfrac{1-x}{\alpha}\right).\end{aligned}$$

Hence, by (15) and (9) we have

$$\mid K_n^\alpha(g,x)-K_n(g,x)\mid\le\left(n^2\cdot\tfrac{\alpha x(1-x)}{1+\alpha}+\sqrt{\tfrac{\alpha x(1-x)}{1+\alpha}}\right)\omega(K_n'(g),n^{-2})_1.$$

Because $x \in [0, 1/n] \cup [1-(1/n), 1]$ and $\alpha = \mathcal{O}(n^{-4})$ we obtain

$$\begin{aligned} |K_n^\alpha(g,x) - K_n(g,x)| &\leq \Big(n^2 \cdot \tfrac{\alpha}{1+\alpha} \cdot \tfrac{1}{n} + \sqrt{\tfrac{\alpha}{1+\alpha} \cdot \tfrac{1}{n}}\,\Big)\omega(K_n'(g), n^{-2})_1 \\ &\leq C \cdot \omega(K_n'(g), n^{-2})_1 \\ &\leq \tfrac{C}{n^2}\|K_n''(g)\|_1 \\ &\leq \tfrac{C}{n^2}\|g''\|_1. \end{aligned}$$

Hence

$$\|K_n^\alpha g - K_n g\|_{L^1([0,1]\setminus E(n))} \leq \tfrac{C}{n^2}\|g''\|_1. \tag{16}$$

Combining (14), (16) and the estimate $\|g'\|_1 \leq C \cdot (\|\varphi^2 g''\|_1 + \|g\|_1)$ of [3, (a), p. 135] we obtain

$$\|K_n^\alpha g - K_n g\|_1 \leq \tfrac{C}{n}(\|g\|_1 + \|\varphi^2 g''\|_1 + \tfrac{1}{n}\|g''\|_1).$$

Then, by Lemma 2

$$\begin{aligned} &\|K_n^\alpha f - K_n f\|_1 \leq \\ &\leq \|K_n^\alpha f - K_n^\alpha g\|_1 + \|K_n^\alpha g - K_n g\|_1 + \|K_n g - K_n f\|_1 \\ &\leq C\|f-g\|_1 + C \cdot (\tfrac{1}{n}\|f\|_1 + \tfrac{1}{n}\|f-g\|_1 + \tfrac{1}{n}\|\varphi^2 g''\|_1 + \tfrac{1}{n^2}\|g''\|_1) \\ &\leq C \cdot (n^{-1}\|f\|_1 + \|f-g\|_1 + \tfrac{1}{n}\|\varphi^2 g''\|_1 + \tfrac{1}{n^2}\|g''\|_1) \end{aligned} \tag{17}$$

for every $g \in C^2[0,1]$. Hence

$$\|K_n^\alpha f - K_n f\|_1 \leq C \cdot (n^{-1}\|f\|_1 + \bar{K}_{2,\varphi}(f, n^{-1})_1).$$

Using the equivalence of $\bar{K}_{2,\varphi}(f,n^{-1})_1$ and $\omega_\varphi^2(f, n^{-1/2})_1$ (see [3, Th. 3.1.1, p. 24]) and [3, Th. 9.3.2, p. 117] we get the assertion of (ii). □

Remark 1. We can establish another direct result which derive from the estimation of $\|K_n^\alpha f - K_n f\|_p$ using the same ideas as in the proof of Theorem 1. More exactly the following result holds. □

Theorem 3. (i) *If* $\alpha = \alpha(n) = \mathcal{O}(n^{-4-\delta})$, $\delta > 0$ *and* $f \in L^p[0,1]$, $1 \leq p \leq \infty$, *then* $\|K_n^\alpha f - f\|_p \leq C\ \omega^2(f, n^{-\delta/2})_p + \|K_n f - f\|_p$;

(ii) *If* $\alpha = \alpha(n) = \mathcal{O}(n^{-8})$ *and* $f \in L^p[0,1]$, $1 \leq p \leq \infty$, *then* $\|K_n^\alpha f - f\|_p \leq C\ \omega_\varphi^2(f, n^{-1})_p + \|K_n f - f\|_p$;

(iii) *If* $\alpha = \alpha(n) = \mathcal{O}(n^{-2})$ *and* $f \in C[0,1]$, $1 \leq p \leq \infty$, *then* $\|K_n^\alpha f - f\|_p \leq C\ \|K_n f - f\|_\infty$;

(iv) *If* $\alpha = \alpha(n) = \mathcal{O}(n^{-1})$ *and* $f \in L^p[0,1]$, $1 < p \leq \infty$, *then* $\|K_n^\alpha f - f\|_p \leq C\ \|K_n f - f\|_p$.

In the above theorem C denotes a constant independent of n and f.

3. A CONVERSE RESULT

In this section we establish the following result

THEOREM 4. *Let* $\alpha = \alpha(n)$ *and* $C_0 \cdot c(p) \cdot \alpha \cdot n \le \gamma < 1$ *for* $n = 1, 2, \ldots,$ *where* C_0 *is the constant from Lemma* 5 *below and* $c(p) = p/(p-1)$. *If* $f \in L^p[0,1]$ *for* $1 < p \le \infty$ *then*

(i) $(1-\gamma)\|K_n f - f\|_p \le \|K_n^\alpha f - f\|_p \le (1+\gamma)\|K_n f - f\|_p$;

(ii) *there exists a constant* $C > 0$ *independent of* n *and* f *such that*

$$C^{-1}[\omega_\varphi^2(f, n^{-\frac{1}{2}})_p + \omega(f, n^{-1})_p] \le \|K_n^\alpha f - f\|_p \le C[\omega_\varphi^2(f, n^{-\frac{1}{2}})_p + \omega(f, n^{-1})_p].$$

Proof of Theorem 4. Again we need a lemma:

LEMMA 5. *For* $f \in L^p[0,1]$, $1 < p \le \infty$ *and* $K_n f$ *defined by* (3), *there exists* C_0 *independent of* n *and* f *such that*

$$\frac{1}{n}\ \|\varphi^2 K_n''(f)\|_p \le C_0 \|K_n f - f\|_p. \tag{18}$$

Proof of Lemma 5. Indeed, in view of [5, Thms 1.1 and 1.2, p. 104] we have $\|K_n f - f\|_p \sim K(f, n^{-1/2})_p$, $1 \le p \le \infty$ and $K(f, n^{-1/2})_p \sim \omega_\varphi^2(f, n^{-1/2})_p + \omega(f, n^{-1})_p$, $1 < p \le \infty$. Then there exists an absolute constant $C > 0$ such that for all $f \in L^p[0,1]$, $1 < p \le \infty$ we have

$$K(f, n^{-\frac{1}{2}})_p \ge C^{-1}[\omega_\varphi^2(f, n^{-\frac{1}{2}})_p + \omega(f, n^{-1})_p] \tag{19}$$

and

$$K(f, n^{-\frac{1}{2}})_p \ \le\ C\ \|K_n f - f\|_p, \tag{20}$$

where $K(f, n^{-1/2})_p$ is defined by (6). Using [3, (9.3.5) and (9.3.7), p. 118] we obtain for $g \in C^2[0,1]$, by (4):

$$\begin{aligned}
\tfrac{1}{n}\|\varphi^2 K_n''(f)\|_p &\le \tfrac{1}{n}(\|\varphi^2 K_n''(f-g)\|_p + \|\varphi^2 K_n''(g)\|_p)\\
&\le \tfrac{C}{n}(n\|f-g\|_p + \|\varphi^2 g''\|_p) =\\
&= C \cdot (\|f-g\|_p + \tfrac{1}{n}\|\varphi^2 g''\|_p).
\end{aligned}$$

Hence, by [3, Th. 2.1.1, p. 11], (19) and (20) we get

$$\begin{aligned}
\tfrac{1}{n}\|\varphi^2 K_n''(f)\|_p &\le C \cdot \omega_\varphi^2(f, n^{-1/2})_p\\
&\le C \cdot (\omega_\varphi^2(f, n^{-1/2})_p + \omega(f, n^{-1})_p)\\
&\le C \cdot K(f, n^{-1/2})_p\\
&\le C \cdot \|K_n f - f\|_p.
\end{aligned}$$

Thus there exists C_0 such that (18) is satisfied. □

Now the proof of the theorem is immediately. By (11) and Lemma 5 we get

$$\begin{aligned} \|K_n f - f\|_p &\leq \|K_n^\alpha f - K_n f\|_p + \|K_n^\alpha f - f\|_p \\ &\leq c(p) \cdot \tfrac{\alpha}{1+\alpha} \cdot \|\varphi^2 K_n''(f)\|_p + \|K_n^\alpha f - f\|_p \\ &\leq C_0 \cdot \tfrac{\alpha}{1+\alpha} \cdot c(p) \cdot n\, \|K_n f - f\|_p + \|K_n^\alpha f - f\|_p \\ &\leq \gamma\, \|K_n f - f\|_p + \|K_n^\alpha f - f\|_p. \end{aligned}$$

Then

$$(1-\gamma)\, \|K_n f - f\|_p \leq \|K_n^\alpha f - f\|_p. \tag{21}$$

Analogously:

$$\|K_n^\alpha f - f\|_p \leq (1+\gamma)\|K_n f - f\|_p. \tag{22}$$

So (21) and (22) imply the conclusion of (i).

For (ii) it is sufficient to observe that (i), (19) and (20) imply the assertion. Thus the theorem is proved. □

REMARK 2. Lemma 5 does not hold for $p = 1$. □

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