REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 31 (2002) no. 2, pp. 171–177
ictp.acad.ro/jnaat

SOME CONCEPTS OF GENERALIZED CONVEX FUNCTIONS (I)

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Abstract. An extension of the concept of convex function is given in a very general framework provided by a set in which a general convexity for its subsets is defined.

MSC 2000. 52A07.
Keywords. Generalized convex functions, generalized partial convex functions.

1. THE NOTIONS OF e-$(\mathcal{H},\mathcal{C})$ CONVEX, a-$(\mathcal{H},\mathcal{C})$ CONVEX, PARTIAL e-$(\mathcal{H},\mathcal{C})$ CONVEX AND PARTIAL a-$(\mathcal{H},\mathcal{C})$ CONVEX FUNCTIONS

A tendency to elaborate enough general theories of convexity for functions to include as much as possible known convexity properties as particular cases arose during the second half of the twentieth century. The generalizations have as a purpose to unify those convexity properties that have similar applications: either in optimization or in shape analysis etc.

In this paper we intend to unify as much as possible convexities for functions from the papers quoted in the references section. This generalized convexity does not use the notion of cone. It might be useful as a classifier for convexity properties of functions in the manner in which a classification for convexities of sets was presented in [3], [4], [5], and [7].

Let W be a nonempty set in which a notion of convexity for sets is defined. By $\mathcal{C} \subseteq 2^W$ we denote the set of all the convex subsets of W.

Let X, Y be two arbitrary nonempty sets and a function $f : M \subseteq X \to Y$. By $\operatorname{gr}(f)$ we shall denote the graph of function f, i.e.

$$\operatorname{gr}(f) = \{(x, f(x)) \,|\, x \in M\}.$$

Let $\mathcal{H}$ be a family of functions $h : 2^{X\times Y} \to 2^W$.

DEFINITION 1. *Function $f : M \subseteq X \to Y$ is called:*

i) *e-$(\mathcal{H},\mathcal{C})$ convex if there is $h \in \mathcal{H}$ such that $h(\operatorname{gr}(f)) \in \mathcal{C}$;*
ii) *a-$(\mathcal{H},\mathcal{C})$ convex if $h(\operatorname{gr}(f)) \in \mathcal{C}$, for each $h \in \mathcal{H}$;*
iii) *partially e-$(\mathcal{H},\mathcal{C})$ convex if there is $P \subseteq M$ and there is $h \in \mathcal{H}$ such that $h(\operatorname{gr}(f|_P)) \in \mathcal{C}$;*

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iv) *partially a-*$(\mathcal{H},\mathcal{C})$ *convex if there is* $P\subseteq M$ *such that* $h(\mathrm{gr}(f|_P))\in\mathcal{C}$, *for each* $h\in\mathcal{H}$.

The "e" letter appearing in the terminology of Definition 1 i) and iii) is used according to its significance of existence from our previous papers [4] and [5]. The "a" letter (from all) is used in Definition 1 ii) and iv) to point the ubiquity, also according to the terminology from [4] and [5].

By analogy, for a set-valued function F defined on $M\subseteq X$ to $2^Y\setminus\emptyset$, we put

$$\mathrm{gr}(F) = \{(x,u)\,|\,x\in M,\ u\in F(x)\}.$$

Definition 2. *Function* $F: M\subseteq 2^X\to(2^Y\setminus\emptyset)$ *is called:*

i) *e-*$(\mathcal{H},\mathcal{C})$ *convex if there is* $h\in\mathcal{H}$ *such that* $h(\mathrm{gr}(F))\in\mathcal{C}$;

ii) *a-*$(\mathcal{H},\mathcal{C})$ *convex if* $h(\mathrm{gr}(F))\in\mathcal{C}$ *, for each* $h\in\mathcal{H}$;

iii) *partially e-*$(\mathcal{H},\mathcal{C})$ *convex if there is* $P\subseteq M$ *and there is* $h\in\mathcal{H}$ *such that* $h(\mathrm{gr}(F|_P))\in\mathcal{C}$;

iv) *partially a-*$(\mathcal{H},\mathcal{C})$ *convex if there is* $P\subseteq M$ *such that* $h(\mathrm{gr}(F|_P))\in\mathcal{C}$, *for each* $h\in\mathcal{H}$.

Obviously, each a-$(\mathcal{H},\mathcal{C})$ convex function is e-$(\mathcal{H},\mathcal{C})$ convex. Also, each partial a-$(\mathcal{H},\mathcal{C})$ convex function is partially e-$(\mathcal{H},\mathcal{C})$ convex. On the other hand, each e-$(\mathcal{H},\mathcal{C})$ convex (a-$(\mathcal{H},\mathcal{C})$ convex) function is partially e-$(\mathcal{H},\mathcal{C})$ convex (partially a-$(\mathcal{H},\mathcal{C})$ convex) function.

2. EXAMPLES OF SCALAR a-$(\mathcal{H},\mathcal{C})$ CONVEX FUNCTIONS

Example 1. ((S,s) convex function). Let Z be a nonempty set, S a nonempty subset of 2^Z, and $s: S\to 2^Z$ a given function. According to [7], a set $A\subseteq Z$ is said to be (S,s) convex if $s(C)\subseteq A$, for each $C\subseteq A$, $C\in S$. □

The notion of (S,s) convex function was defined in [2] by means of the notion of epigraph in a particular framework, which will be described as follows. T is supposed to be a totally ordered nonempty set, denoting its order relation by $<$. Symbol $x\le y$ denotes the situation when either $x<y$ or $x=y$, $x,y\in T$. Let X be a nonempty set. We suppose that a (S,s) property of convexity is given in $X\times T$.

For a function $f: A\subseteq X\to T$ we define the epigraph by the set

$$\mathrm{epi}(f) = \{(x,t)\,|\,f(x)\le t\}\subseteq A\times T.$$

Definition 3. *A function* $f: X\to T$ *is said to be* (S,s) *convex if* epi(f) *is* (S,s) *convex in* $X\times T$.

If we take $Y = T$, $W = X\times T$, $\mathcal{H} = \{h\}$, where the map $h: 2^{X\times Y}\to 2^W$ is given by

$$h(M) = \{(x,u)\in X\times T \mid \exists w\in T, \text{with}\,(x,w)\in M, \text{s. t.}\, w\le u\}$$

for each $M\subseteq X\times T$, $M\neq\emptyset$, and $h(\emptyset)=\emptyset$, and $\mathcal{C}$ is the collection of all (S,s) convex sets in X, then it is easy to prove that:

THEOREM 4. *Function* $f : A \subseteq X \to T$ *is* (S, s) *convex if and only if* f *is a-*$(\mathcal{H}, \mathcal{C})$ *convex.*

Proof. It is easy to see that

$$h(\mathrm{gr}(f)) \;=\; \mathrm{epi}(f). \tag{1}$$

Then we have $h(\mathrm{gr}(f)) \in \mathcal{C}$ if and only if the set $\mathrm{epi}(f)$ is (S, s) convex. Hence, function f is a-$(\mathcal{H}, \mathcal{C})$ convex if and only if it is (S, s) convex. □

REMARK 1. Because a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex in classical sense if and only if it is (S, s) convex, where

$$S = \{\{(x, y),\, (u, v)\} \,|\, x, u \in \mathbb{R}^n,\, y, v \in \mathbb{R}\}$$

and $s : S \to 2^{\mathbb{R}^n \times \mathbb{R}}$,

$$s(\{(x, y),\, (u, v)\}) \;=\; \{((1-\lambda)x + \lambda u,\, (1-\lambda)y + \lambda v) \,|\, \lambda \in [0, 1]\},$$

for each $\{(x, y),\, (u, v)\} \in S$, the classical convexity for functions is an a-$(\mathcal{H}, \mathcal{C})$ convexity. □

EXAMPLE 2. (Weakly-convex function) Let X be a real linear space and let A be a nonempty subset of X. We remember (see [1]) that a function $f : A \to \mathbb{R}$ is said to be weakly-convex if for each $x,\, y \in A$ there is a $p \in]0, 1[$, such that

$$(1-p)x + py \in A \tag{2}$$

and

$$f\big((1-p)x + py\big) \le (1-p)f(x) + pf(y). \tag{3}$$

□

Let us take $X = X$, $Y = \mathbb{R}$, $W = X \times \mathbb{R}$, $\mathcal{H} = \{h\}$, where the map $h : 2^{X \times \mathbb{R}} \to 2^{X \times \mathbb{R}}$ is defined by

$$h(M) \;=\; \{(x, u) \,|\, x \in X,\; u \in \mathbb{R},\; u \ge inf\{w \,|\, (x, w) \in M\}\},$$

for each $M \subseteq X \times \mathbb{R}$, $M \neq \emptyset$, and by $h(M) = \emptyset$, if $M = \emptyset$.

In $2^{X \times \mathbb{R}}$ we consider the set

$$S \;=\; \{\, \{(x, u), (y, v)\} \;|\; x, y \in X \text{ and } u, v \in \mathbb{R}\}.$$

Let

$$\mathcal{S} \;=\; \{s_p \,|\, p \in]0, 1[\},$$

where $s_p : S \to 2^{X \times \mathbb{R}}$ is the map defined for each $\{(x, u),\, (y, v)\} \in S$ by

$$s_p(\{(x, u),\, (y, v)\}) \;=\; \{((1-p)x + py,\, (1-p)u + pv)\}.$$

In accordance with [7] a subset U of $X \times \mathbb{R}$ is $(S, \mathcal{S})-$ convex if for each $C \in S$, with $C \subseteq U$, there is $s \in \mathcal{S}$ such that $s(C) \subseteq U$.

If we denote by $\mathcal{C}$ the collection of all $(S, \mathcal{S})$ convex subsets of $X \times \mathbb{R}$, we obtain:

THEOREM 5. *Function* $f : A \subseteq X \to \mathbb{R}$ *is weakly-convex if and only if* f *is a-*$(\mathcal{H}, \mathcal{C})$ *convex.*

Proof. Necessity. Let $f : A \to \mathbb{R}$ be a weakly-convex function and let $C \in S$, $C \subseteq h(\mathrm{gr}(f))$. Then there exist (x,u), $(y,v) \in X \times \mathbb{R}$ such that $C = \{(x,u),\,(y,v)\}$ and $\{(x,u),\,(y,v)\} \subseteq h(\mathrm{gr}(f))$. From the definitions of $\mathrm{gr}(f)$ and h, we get that $x \in A$, $y \in A$. Also, as $(x, f(x)) \in \mathrm{gr}(f)$ and $(y, f(u)) \in \mathrm{gr}(f)$, it follows that

$$u \geq f(x) \quad \text{and} \quad v \geq f(y). \tag{4}$$

On the other hand, since f is weakly convex, there is $p \in]0,1[$ such that (2) and (3) are true. Then, from (4), it results

$$(1-p)u + pv \geq (1-p)f(x) + pf(y). \tag{5}$$

(2), (5) and (3) imply that

$$(1-p)x + py \in A, \quad (1-p)u + pv \geq f\big((1-p)x + py\big),$$

i.e., $((1-p)x + py,\,(1-p)u + pv) \in h(\mathrm{gr}(f))$, or $s_p(C) \subseteq h(\mathrm{gr}(f))$. Hence, set $h(\mathrm{gr}(f))$ is $(S, \mathcal{S})$-convex. Therefore f is an a-$(\mathcal{H}, \mathcal{C})$ convex function.

Sufficiency. Let now f be a-$(\mathcal{H}, \mathcal{C})$ convex, and let x, $y \in A$. Because $(x, f(x)) \in \mathrm{gr}(f)$ and $(y, f(u)) \in \mathrm{gr}(f)$, if we take $C = \{(x,u),\,(y,v)\}$, then we have $C \in S$ and $C \subseteq h(\mathrm{gr}(f))$. Since f is a-$(\mathcal{H}, \mathcal{C})$ convex, we get that there exists $s \in \mathcal{S}$ such that

$$s(C) \subseteq h(\mathrm{gr}(f)). \tag{6}$$

As $s \in \mathcal{S}$, there is $p \in]0,1[$ such that $s = s_p$. Then

$$s(C) = s_p(C) = \{((1-p)x + py,\,(1-p)f(x) + pf(y))\} \subseteq h(\mathrm{gr}(f)).$$

This implies $((1-p)x + py \in A$, and $(1-p)f(x) + pf(y) \geq f((1-p)x + py)$. Therefore, we conclude that the function f is weakly-convex, as x and y was arbitrarily chosen in A. □

Example 3. (Convex functions of order n). Let be n a natural number. By $\mathcal{P}_{n+1}$ we denote the set of all algebraic polynomials of degree $n+1$ or less, having real coefficients.

Taking into account the representation of polynomials of degree $n+1$ or less as a linear combination of the elements of a basis of the real linear space $\mathcal{P}_{n+1}$,

$$\sum_{i=0}^{n+1} a_i x_i, \quad \text{with}\, a_i \in \mathbb{R},\, i \in \{0,1,2,...,n+1\},$$

we obtain the partition

$$\mathcal{P}_{n+1} \;=\; \mathcal{P}^{-}_{n+1} \;\cup\; \mathcal{P}_n \;\cup\; \mathcal{P}^{+}_{n+1},$$

where $\mathcal{P}^{-}_{n+1}$ ($\mathcal{P}^{+}_{n+1}$) denotes the set of all the polynomials from $\mathcal{P}_{n+1}$ that have the coefficient $a_{n+1} < 0$ (respectively $a_{n+1} > 0$), and $\mathcal{P}_n$ is the set of all polynomials of degree n. □

Let A be a subset of $\mathbb{R}$, with $|A| \geq n+2$, and let

$$x_1,\, x_2, ..., x_{n+2} \tag{7}$$

be a system of $n+2$ distinct points from A.

Definition 6. (*see* [10]). *A function* $f : A \to \mathbb{R}$ *is called*

i) *convex of order* n *on points* (7) *if* $L_n(x_1, ..., x_{n+1}; f) \in \mathcal{P}^+_{n+1}$, *where* $L_n(x_1, ..., x_{n+1}; f)$ *denotes the Lagrange interpolation polynomial.*
ii) *concave of order* n *on points* (7) *if* $L_n(x_1, ..., x_{n+1}; f) \in \mathcal{P}^-_{n+1}$.
iii) *convex of order* n *on* A *if* f *is convex of order* n *on each system of* $n+2$ *points of type* (7) *from* A.
iv) *concave of order* n *on* A *if* f *is concave of order* n *on each system of* $n+2$ *points of type* (7) *from* A.

We can prove that the convexity of order n is also an a-$(\mathcal{H}, \mathcal{C})$ convexity. For this we introduce the notion of convexity of order n for sets. Let D a nonempty subset of $\mathbb{R}$, and $g : D \to \mathbb{R}$ a given function.

Definition 7. *A subset* C *of* $\mathbb{R}^2$ *is said to be convex of order* n *with respect to function* g *if we have*

$$(x_{n+2}, L_n(x_1, ..., x_{n+1}; f)(x_{n+2})) \in \mathbb{R}^2 \setminus (C \cup \mathrm{gr}(f)), \tag{8}$$

for each system of $n+2$ *points* $x_1, ..., x_{n+2}$ *from* D, *with*

$$x_1 < ... < x_{n+1} < x_{n+2}.$$

In the following by $\mathcal{C}_{n,g}$ we denote the set of all subsets C of $\mathbb{R}^2$, which are convex of order n with respect to g.

Let us take $X = Y = \mathbb{R}$, $W = \mathbb{R}^2$, $\mathcal{H} = \{h\}$, where the map $h : 2^{X\times Y} \to 2^W$ is given by

$$h(M) = \{(x, u) \,|\, x \in pr_1(M), \text{ and } u \in \mathbb{R},\ u \geq inf\{w \,|\, (x, w) \in M\}\},$$

for each $M \subseteq \mathbb{R}^2$, $M \neq \emptyset$, and $h(\emptyset) = \emptyset$.

Theorem 8. *If* n *is a natural number and* A *is a subset of* $\mathbb{R}$, *with* $|A| \geq n+2$, *then a function* $f : A \to \mathbb{R}$ *is convex of order* n *if and only if it is a-*$(\mathcal{H}, \mathcal{C}_{n,f})$ *convex.*

Proof. In our proof we use the following property of convex function of n order (see [9]):

Lemma 9. *Function* $f : A \to \mathbb{R}$ *is convex of order* n *if and only if*

$$f(x_{n+2}) - L_n(x_1, ..., x_{n+1}; f)(x_{n+2}) > 0, \tag{9}$$

for all system of points

$$x_1 < x_2 < ... < x_{n+2} \quad from\ \ A. \tag{10}$$

Proof. We remark that

$$h(\mathrm{gr}(f)) = \{(x,y) \in A \times \mathbb{R} \mid y \geq f(x)\} = \mathrm{epi}(f). \tag{11}$$

Necessity. Let us suppose that function f is convex of order n on A. Then, according to lemma, we have (9), for all system of points (10). It means that

$$f(x_{n+2}) > L_n(x_1, ..., x_{n+1}; f)(x_{n+2}),$$

for all points $x_1 < x_2 < ... < x_{n+2}$ from A. Therefore

$$(x_{n+2}, L_n(x_1, x_2, ..., x_{n+1}; f)(x_{n+2})) \notin \mathrm{epi}(f),$$

for all points $x_1 < x_2 < ... < x_{n+2}$ from A, i.e.,

$$(x_{n+2}, L_n(x_1, x_2, ..., x_{n+1}; f)(x_{n+2})) \in (\mathbb{R}^2) \setminus \mathrm{epi}(f),$$

for all points $x_1 < x_2 < ... < x_{n+2}$ from A,

As a consequence of (11), the set $h(\mathrm{gr}(f))$ is convex of order n with respect to f. Hence, function f is a-$(\mathcal{H}, \mathcal{C}_{n,f})$ convex.

Sufficiency. Now, as function f is a-$(H, \mathcal{C}_{f,n})$ convex, from the statement that $h(\mathrm{gr}(f)) \in \mathcal{C}_{f,n}$, i.e. $\mathrm{epi}(f) \in \mathcal{C}_{f,n}$, we obtain that

$$(x_{n+2}, L_n(x_1, x_2, ..., x_{n+1}; f)(x_{n+2})) \notin \mathrm{epi}(f)$$

for all points $x_1 < x_2 < ... < x_{n+2}$ from A. It means that

$$f(x_{n+2}) > L_n(x_1, x_2, ..., x_{n+1}; f)(x_{n+2})$$

for all $x_1 < x_2 < ... < x_{n+2}$ from A. So, in view of lemma, function f is convex of order n on A. $\square$

Analogously we obtain the following corollary.

Corollary 10. *A function $f : A \to \mathbb{R}$ is concave of order n on A if and only if the function $g = -f$ is a-$(\mathcal{H}, \mathcal{C}_{n,f})$ convex.*

We remark that a subset C of $\mathbb{R}^2$ is convex of order n with respect to a given function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ if and only if C is $(e,a) - ((S,s),r)$ convex, where $S = 2^{\mathbb{R}^2}$, $s : 2^{\mathbb{R}^2} \to 2^{\mathbb{R}^2}$ is the map given by

$$s(U) = \{(x_{n+2}, L_n(x_1, x_2, ..., x_{n+1}; f)(x_{n+2})) \,|\, x_1 < x_2 < ... < x_{n+2},\ x_i \in A\},$$

for each $U \subseteq \mathbb{R}^2$, and $r : 2^{\mathbb{R}^2} \to 2^{\mathbb{R}^2}$ is the map given by

$$r(U) = \mathbb{R}^2 \setminus (U \cup \mathrm{gr}(f)), \quad \text{for each}\, U \subseteq \mathbb{R}^2.$$

The case of convex functions of order n with respect to an interpolating set (E. Popoviciu, 1972) can be treated in the same manner.

Also, we can model in the same way the strong convex mappings with respect to a given set from [6].

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Received by the editors: March 7, 2002.