FUNCTIONS WITH BOUNDED $E$-$d$-VARIATION ON UNDIRECTED TREE NETWORKS

DANIELA MARIAN

Abstract. In this paper we define and study functions with bounded $E$-$d$-variation on undirected tree networks. For these functions with bounded $E$-$d$-variation we establish a Jordan type theorem. We adopt the definition of network as metric space introduced by P. M. Dearing and R. L. Francis (1974).

MSC 2000. 65P40.

Keywords. Networks, bounded $E$-$d$-variation.

1. INTRODUCTION

In [9] are introduced a class of sets and a class of functions called $E$-convex sets and $E$-convex functions. This kind of generalized convexity is based on the effect of an operator $E$ on the sets and domain the definition of the functions. In [1] are defined and studied $E$-monotone functions and functions of bounded $E$-variation.

In the following lines we will define and study $E$-$d$-monotone functions and functions with bounded $E$-$d$-variation on undirected tree networks.

We recall first the definitions of undirected networks as metric space introduced in [2] and also used in many other papers (see, e.g., [3], [5], etc.).

We consider an undirected, connected graph $G = (W,A)$, without loops or multiple edges. To each vertex $w_i \in W = \{w_1, \ldots, w_m\}$ we associate a point $v_i$ from an euclidean space $X$. This yields a finite subset $V = \{v_1, \ldots, v_m\}$ of $X$, called the vertex set of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset X$ called edge of the network. We assume that any two edges have no interior common points. Consider that $[v_i, v_j]$ has the positive length $l_{ij}$ and denote by $U$ the set of all edges. We define the network $N = (V,U)$ by

$$N = \{x \in X \mid \exists (w_i, w_j) \in A \text{ such that } x \in [v_i, v_j] \}.$$ 

It is obvious that $N$ is a geometric image of $G$, which follows naturally from an embedding of $G$ in $X$. Suppose that for each $[v_i, v_j] \in U$ there exist a continuous one-to-one mapping $\theta_{ij} : [v_i, v_j] \to [0,1]$ with

$$\theta_{ij}(v_i) = 0,$$

$$\theta_{ij}(v_j) = 1,$$

**St. Rășinari 5, ap. 28, Cluj-Napoca, Romania, e-mail: danielamarian@personal.ro.**
and

\[ \theta_{ij}([v_i, v_j]) = [0, 1]. \]

We denote by \( T_{ij} \) the inverse function of \( \theta_{ij} \).

Any connected and closed subset of an edge bounded by two points \( x \) and \( y \) of \([v_i, v_j]\) is called a closed subedge and is denoted by \([x, y]\). If one or both of \( x, y \) are missing we say than the subedge is open in \( x \), or in \( y \) or is open and we denote this by \((x, y]\), \([x, y)\) or \((x, y)\), respectively. Using \( \theta_{ij} \), it is possible to compute the length of \([x, y]\) as

\[ l([x, y]) = |\theta_{ij}(x) - \theta_{ij}(y)| \cdot l_{ij}. \]

Particularly we have

\[ l([v_i, v_j]) = l_{ij}, \]

\[ l([v_i, x]) = \theta_{ij}(x) \cdot l_{ij} \]

and

\[ l([x, v_j]) = (1 - \theta_{ij}(x)) \cdot l_{ij}. \]

A path \( L(x, y) \) linking two points \( x \) and \( y \) in \( N \) is a sequence of edges and at most two subedges at extremities, starting at \( x \) and ending at \( y \). If \( x = y \) then the path is called cycle. The length of a path (cycle) is the sum of the lengths of all its component edges and subedges and will be denoted by \( l(L(x, y)) \). If a path (cycle) contains only distinct vertices then we call it elementary.

A network is connected if for any points \( x, y \in N \) there exists a path \( L(x, y) \subset N \).

A connected network without cycles is called tree. In a tree network \( N \) there is an unique path between two points \( x, y \in N \).

Let \( L^*(x, y) \) be a shortest path between the points \( x, y \in N \). This path is also called geodesic.

**Definition 1.** [2] For any \( x, y \in N \), the distance from \( x \) to \( y \), \( d(x, y) \), in the network \( N \) is the length of a shortest path from \( x \) to \( y \):

\[ d(x, y) = l(L^*(x, y)). \]

It is obvious that \((N, d)\) is a metric space.

For \( x, y \in N \), we denote

\[ (x, y) = \{ z \in N \mid d(x, z) + d(z, y) = d(x, y) \}, \]

and \((x, y)\) is called the metric segment between \( x \) and \( y \).

**Definition 2.** [2] A set \( D \subset N \) is called \( d \)-convex if

\( (x, y) \subset D \), for all \( x, y \in D \).

We consider now a map \( E : N \to N \).

**Definition 3.** [7] A set \( M \subset N \) is said to be \( E \)-\( d \)-convex if

\[ (E(x), E(y)) \subset M, \] for each \( x, y \in M \).
Theorem 4. [7] If a set \( M \subset N \) is \( E\)-convex then \( E(M) \subseteq M \).

Theorem 5. [7] If \( E(M) \) is \( d\)-convex and \( E(M) \subseteq M \) then \( M \) is \( E\)-convex.

2. \( E\)-MONOTONE FUNCTIONS ON UNDIRECTED TREE NETWORKS

We consider an undirected tree network \( N = (V, U) \), a map \( E : N \to N \) and two points \( x, y \in N \).

We define the following order relation on \( \langle x, y \rangle_E \).

\[
\langle x, y \rangle_E = \{ z \in N \mid d(E(x), E(y)) = d(E(x), E(z)) + d(E(z), E(y)) \}.
\]

Obviously \( E(\langle x, y \rangle_E) \subseteq \langle E(x), E(y) \rangle \). Generally, the converse inequality is not true.

Now, let us define the following order relation on \( \langle x, y \rangle_E \). For \( z_1, z_2 \in \langle x, y \rangle_E \) with

\[
d(E(x), E(z_1)) = \alpha_1 d(E(x), E(y))
\]

and

\[
d(E(x), E(z_2)) = \alpha_2 d(E(x), E(y)),
\]

then we shall write \( z_1 = E z_2 \). If two points \( z_1, z_2 \in \langle x, y \rangle_E \) satisfy the equality \( E(z_1) = E(z_2) \) then we shall write \( z_1 = E z_2 \). For every \( z \in \langle x, y \rangle_E \) we have \( x \leq_E y \).

We consider now the function \( f : N \to \mathbb{R} \) and the points \( x, y \in N \).

Definition 6. (1) The function \( f : N \to \mathbb{R} \) is said to be \( E\)-d-increasing between \( x \) and \( y \) if for every \( z_1, z_2 \in \langle x, y \rangle_E \) such that \( z_1 \leq_E z_2 \) we have

\[
f(E(z_1)) \leq f(E(z_2)).
\]

(2) The function \( f : N \to \mathbb{R} \) is said to be \( E\)-d-decreasing between \( x \) and \( y \) if for every \( z_1, z_2 \in \langle x, y \rangle_E \) such that \( z_1 \leq_E z_2 \) we have

\[
f(E(z_1)) \geq f(E(z_2)).
\]

(3) The function \( f : N \to \mathbb{R} \) is said to be \( E\)-d-constant between \( x \) and \( y \) if

\[
f(E(x)) = f(E(z)), \quad \text{for every } z \in \langle x, y \rangle_E.
\]

A function that is either \( E\)-d-increasing or \( E\)-d-decreasing between \( x \) and \( y \) is said to be \( E\)-d-monotone between \( x \) and \( y \). If all the inequalities in Definition 6 are strict then \( f \) is called strictly \( E\)-d-increasing, strictly \( E\)-d-decreasing or strictly \( E\)-d-monotone.

Remarks. 1. If a function \( f : N \to \mathbb{R} \) is both \( E\)-d-increasing and \( E\)-d-decreasing between \( x \) and \( y \) then it is \( E\)-d-constant between \( x \) and \( y \).

2. If a function \( f : N \to \mathbb{R} \) is \( E\)-d-increasing between \( x \) and \( y \) then it is \( E\)-d-decreasing between \( y \) and \( x \).

In the following we give an example of \( E\)-d-monotone function.
Example 1. We consider a tree network \( N = (V, U) \) with \( V = \{v_1, v_2, v_3\} \) and \( U = \{[v_1, v_2], [v_1, v_3]\} \) such that \( l([v_1, v_2]) = 1 \) and \( l([v_1, v_3]) \geq 1 \). For every edge \([v_i, v_j]\) in \( U \) we consider the corresponding function \( \theta_{ij} : [v_i, v_j] \to [0, 1] \). For every \( z \in [v_i, v_j] \) we denote by \( z' \) the point of the edge \([v_1, v_2]\) such that \( d(v_1, z') = \theta_{ij}(z) \) and let \( l_z = \theta_{ij}(z) \). We define now \( E : N \to N, \)

\[ E(z) = z', \quad \forall z \in N. \]

For the points \( x = v_1 \) and \( y = v_2 \) we have

\[ \langle x, y \rangle_E = N \quad \text{and} \quad E(\langle x, y \rangle_E) = [v_1, v_2]. \]

The function \( f : N \to \mathbb{R}, \)

\[ f(z) = l_z, \quad \forall z \in N \]

is \( E\)-d-increasing between \( x \) and \( y \). Indeed, if \( z_1, z_2 \in \langle x, y \rangle_E \) and \( z_1 \leq_E z_2 \) then \( l_{z_1} \leq l_{z_2} \) and consequently \( f(z_1) \leq f(z_2) \). \( \square \)

3. Functions with bounded \( E\)-d-variation on undirected tree networks

We consider an undirected tree network \( N = (V, U) \), a map \( E : N \to N \), an \( E\)-d-convex set \( M \subset N \) and the function \( f : M \to \mathbb{R} \). We also consider the points \( x, y \in M \).

For a division \((\sigma)\) of the set \( \langle x, y \rangle_E \cap M \) by the points

\[ x =_E z_0 \prec_E z_1 \prec_E z_2 \prec_E \cdots \prec_E z_q =_E y \]

we define the number

\[ \sqrt{E; f, \sigma} = \sum_{i=1}^{q} |f(E(z_i)) - f(E(z_{i-1}))|, \]

called the \( E\)-d-variation of the function \( f \) on the division \((\sigma)\).

The function \( f : M \to \mathbb{R} \) is said to be with bounded \( E\)-d-variation on \( \langle x, y \rangle_E \cap M \) if

\[ \sqrt{E; f, \sigma} < \infty. \]

Definition 7. The number

\[ \sqrt{E; f, \sigma} = \sup \left\{ \sqrt{E; f, \sigma} \mid \sigma \in D \right\} \]

is called the total \( E\)-d-variation of the function \( f \) on \( \langle x, y \rangle_E \cap M \).

Definition 8. The function \( f : M \to \mathbb{R} \) is said to be with bounded \( E\)-d-variation on \( \langle x, y \rangle_E \cap M \) if

\[ \sqrt{E; f, \sigma} < \infty. \]

Theorem 9. If the function \( f : M \to \mathbb{R} \) is \( E\)-d-monotone between \( x \) and \( y \) then it is with bounded \( E\)-d-variation on \( \langle x, y \rangle_E \cap M \).
Proof. If the function $f : M \to \mathbb{R}$ is $E$-d-increasing between $x$ and $y$ then for every division \([2]\)

\[
\sqrt{(E; f, \sigma)} = \sum_{i=1}^{q} \left| f(E(z_i)) - f(E(z_{i-1})) \right|
\]

\[
= \sum_{i=1}^{q} \left[ f(E(z_i)) - f(E(z_{i-1})) \right]
\]

\[
= f(E(y)) - f(E(x)),
\]

and therefore

\[
\sqrt{(E; f)} = f(E(y)) - f(E(x)) < \infty.
\]

If the function is $E$-d-decreasing between $x$ and $y$ then the proof is analogously.

\[\square\]

Corollary 10. If the function $f : M \to \mathbb{R}$ is $E$-d-constant between $x$ and $y$ then it is with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$ and the total $E$-d-variation of $f$ on $\langle x, y \rangle_E \cap M$ is zero.

We denote by $E_1$ the restriction of the map $E : N \to \mathbb{R}$ to the $E$-d-convex set $M$. Since the set $M$ is $E$-d-convex, $E(M) \subseteq M$.

Theorem 11. If the function $f : M \to \mathbb{R}$ is with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$ then the function $f \circ E_1 : M \to \mathbb{R}$ is bounded on $\langle x, y \rangle_E \cap M$.

Proof. For the particular division $\sigma = (x < E z < E y)$ of $\langle x, y \rangle_E \cap M$ we have

\[
\sqrt{(E; f, \sigma)} = \left| f(E(z)) - f(E(x)) \right| + \left| f(E(y)) - f(E(z)) \right| \leq \sqrt{(E; f)}
\]

and

\[
\left| f(E(z)) \right| \leq \left| f(E(z)) - f(E(x)) \right| + \left| f(E(x)) \right| \leq \sqrt{(E; f)} + \left| f(E(x)) \right|.
\]

Consequently the function $f \circ E_1 : M \to \mathbb{R}$ is bounded on $\langle x, y \rangle_E \cap M$.

\[\square\]

The following two theorems are immediately implied.

Theorem 12. If the functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$ then the functions $f + g$, $f - g$, and $fg$ are with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$.

Theorem 13. If the functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$ and there is a number $\eta > 0$ such that

\[
g(E(z)) \geq \eta, \quad \text{for every } z \in \langle x, y \rangle_E \cap M,
\]

then the function $f / g$ is with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$. 

**Definition 14.** The function $f : M \to \mathbb{R}$ satisfies the $E$-Lipschitz condition on $\langle x, y \rangle_E \cap M$ if there is a number $k > 0$ such that for any pair of points $z_1, z_2 \in \langle x, y \rangle_E \cap M$ it is satisfied the relation:

$$|f(E(z_1)) - f(E(z_2))| \leq kd(E(z_1), E(z_2)).$$

**Theorem 15.** If the function $f : M \to \mathbb{R}$ satisfies the $E$-Lipschitz condition on $\langle x, y \rangle_E \cap M$ then it is with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$.

**Proof.** Indeed, if the function $f : M \to \mathbb{R}$ satisfies the $E$-Lipschitz condition on $\langle x, y \rangle_E \cap M$ then there is a number $k > 0$ such that for any pair of points $z_1, z_2 \in \langle x, y \rangle_E \cap M$ is satisfied the relation (3). Consequently we have:

$$\sqrt{\left(\sum_{i=1}^{q} |f(E(z_i)) - f(E(z_{i-1}))| \right)} \leq \sum_{i=1}^{q} kd(E(z_i), E(z_{i-1})) \leq kd(E(x), E(y)) < \infty$$

and hence $\sqrt{\left(\sum_{i=1}^{q} |f(E(z_i)) - f(E(z_{i-1}))| \right)} < \infty$ and $f$ is with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$.

We consider now a point $z \in \langle x, y \rangle_E \cap M \setminus \{t \in M \mid t_E = x \text{ or } t_E = y\}$.

**Theorem 16.** If the function $f : M \to \mathbb{R}$ is with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$ then it is with bounded $E$-d-variation on $\langle x, z \rangle_E \cap M$ and on $\langle z, y \rangle_E \cap M$ and

$$\sqrt{\left(\sum_{i=1}^{q} |f(E(z_i)) - f(E(z_{i-1}))| \right)} = \sqrt{\left(\sum_{i=1}^{q} |f(E(z_i)) - f(E(z_{i-1}))| \right)} + \sqrt{\left(\sum_{i=1}^{q} |f(E(z_i)) - f(E(z_{i-1}))| \right)}$$

**Theorem 17.** If the function $f : M \to \mathbb{R}$ is with bounded $E$-d-variation on $\langle x, z \rangle_E \cap M$ and on $\langle z, y \rangle_E \cap M$ then it is with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$.

In the following lines we will establish a Jordan type theorem.

**Theorem 18.** The function $f : \langle x, y \rangle_E \cap M \to \mathbb{R}$ is with bounded $E$-d-variation on $\langle x, y \rangle_E \cap M$ if and only if there exist two $E$-d-increasing functions between $x$ and $y$, $g : \langle x, y \rangle_E \cap M \to \mathbb{R}$ and $h : \langle x, y \rangle_E \cap M \to \mathbb{R}$ such that $f = g - h$.

**Proof.** The sufficiency of the condition follows from Theorem [9] and Theorem [12].
For the necessity, let us define the function

\[
g : (x, y) \in E \cap M \to \mathbb{R}, \quad g(z) = \begin{cases} \frac{z}{x} (E; f), & \text{if } x < E z \leq E y \\ 0, & \text{if } x = E z. \end{cases}
\]

From Theorem 16 follows that the function \( g \) is \( E \)-d-increasing between \( x \) and \( y \).

We define now the function

\[
h : (x, y) \in E \cap M \to \mathbb{R}, \quad h(z) = g(z) - f(z).
\]

This function is \( E \)-d-increasing between \( x \) and \( y \). Indeed, if we consider the points \( z', z'' \in (x, y) \in E \cap M \), such that \( z' < E z'' \), we have

\[
h(z'') = g(z'') - f(z'') = g(z') + \frac{z''}{z'} (E; f) - f(z''),
\]

\[
h(z'') - h(z') = \frac{z''}{z'} (E; f) - [f(z'') - f(z')].
\]

But

\[
f(z'') - f(z') \leq \frac{z''}{z'} (E; f).
\]

Consequently \( h(z'') - h(z') \geq 0 \), that is, \( h \) is \( E \)-d-increasing between \( x \) and \( y \). Hence \( f = g - h \), where the functions \( g \) and \( h \) are \( E \)-d-increasing between \( x \) and \( y \).

**Remark 1.** The representation of a function with bounded \( E \)-d-variation as a difference of two \( E \)-d-increasing functions is not unique. Indeed, if \( f = g - h \) and the functions \( g \) and \( h \) are \( E \)-d-increasing between \( x \) and \( y \) then we also have

\[
f = g + c - (h + c),
\]

\( c \) being constant on \((x, y) \in E \cap M \). The functions \( g + c \) and \( h + c \) are \( E \)-d-increasing between \( x \) and \( y \) too.

**Remark 2.** In [6] we already defined and studied another class of functions with bounded variation on undirected networks.

**REFERENCES**


Received by the editors: January 16, 2001.