REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 31 (2002) no. 2, pp. 179–186
ictp.acad.ro/jnaat

FUNCTIONS WITH BOUNDED E-d-VARIATION ON UNDIRECTED TREE NETWORKS

DANIELA MARIAN*

Abstract. In this paper we define and study functions with bounded E-d-variation on undirected tree networks. For these functions with bounded E-d-variation we establish a Jordan type theorem. We adopt the definition of network as metric space introduced by P. M. Dearing and R. L. Francis (1974).

MSC 2000. 65P40.
Keywords. Networks, bounded E-d-variation.

1. INTRODUCTION

In [9] are introduced a class of sets and a class of functions called E-convex sets and E-convex functions. This kind of generalized convexity is based on the effect of an operator E on the sets and domain the definition of the functions. In [1] are defined and studied E-monotone functions and functions of bounded E-variation.

In the following lines we will define and study E-d-monotone functions and functions with bounded E-d-variation on undirected tree networks.

We recall first the definitions of undirected networks as metric space introduced in [2] and also used in many other papers (see, e.g., [3], [5], [4], etc.).

We consider an undirected, connected graph $G = (W, A)$, without loops or multiple edges. To each vertex $w_i \in W = \{w_1, \ldots, w_m\}$ we associate a point v_i from an euclidean space X. This yields a finite subset $V = \{v_1, \ldots, v_m\}$ of X, called the vertex set of the network. We also associate to each edge $(w_i, w_j) \in A$ a rectifiable arc $[v_i, v_j] \subset X$ called edge of the network. We assume that any two edges have no interior common points. Consider that $[v_i, v_j]$ has the positive length l_{ij} and denote by U the set of all edges. We define the network $N = (V, U)$ by

$$N = \{x \in X \mid \exists\, (w_i, w_j) \in A \text{ such that } x \in [v_i, v_j]\,\}.$$

It is obvious that N is a geometric image of G, which follows naturally from an embedding of G in X. Suppose that for each $[v_i, v_j] \in U$ there exist a continuous one-to-one mapping $\theta_{ij} : [v_i, v_j] \to [0, 1]$ with

$$\theta_{ij}\,(v_i) = 0,$$

$$\theta_{ij}\,(v_j) = 1,$$

*St. Răşinari 5, ap. 28, Cluj-Napoca, Romania, e-mail: danielamarian@personal.ro.

and

$$\theta_{ij}\left([v_i, v_j] \right) = [0,1] .$$

We denote by T_{ij} the inverse function of θ_{ij}.

Any connected and closed subset of an edge bounded by two points x and y of $[v_i, v_j]$ is called a closed subedge and is denoted by $[x, y]$. If one or both of x, y are missing we say than the subedge is open in x, or in y or is open and we denote this by $(x, y]$, $[x, y)$ or (x, y), respectively. Using θ_{ij}, it is possible to compute the length of $[x, y]$ as

$$l\left([x,y] \right) = \left| \theta_{ij}\left(x\right) - \theta_{ij}\left(y\right) \right| \cdot l_{ij}.$$

Particularly we have

$$l\left([v_i, v_j]\right) = l_{ij},$$
$$l\left([v_i, x]\right) = \theta_{ij}\left(x\right) l_{ij}$$

and

$$l\left([x, v_j] \right) = \left(1 - \theta_{ij}\left(x\right) \right) \cdot l_{ij}.$$

A path $L(x, y)$ linking two points x and y in N is a sequence of edges and at most two subedges at extremities, starting at x and ending at y. If $x = y$ then the path is called cycle. The length of a path (cycle) is the sum of the lengths of all its component edges and subedges and will be denoted by $l\left(L\left(x, y\right)\right)$. If a path (cycle) contains only distinct vertices then we call it elementary.

A network is connected if for any points $x, y \in N$ there exists a path $L\left(x, y\right) \subset N$.

A connected network without cycles is called tree. In a tree network N there is an unique path between two points $x, y \in N$.

Let $L^*\left(x, y\right)$ be a shortest path between the points $x, y \in N$. This path is also called geodesic.

Definition 1. [2]. *For any $x, y \in N$, the distance from x to y, $d\left(x, y\right)$, in the network N is the length of a shortest path from x to y:*

$$d\left(x, y\right) = l\left(L^*\left(x, y\right)\right).$$

It is obvious that (N, d) is a metric space.

For $x, y \in N$, we denote

$$\langle x, y\rangle = \left\{ z \in N \mid d\left(x, z\right) + d\left(z, y\right) = d\left(x, y\right) \right\}, \tag{1}$$

and $\langle x, y\rangle$ is called the metric segment between x and y.

Definition 2. [2]. *A set $D \subset N$ is called d-convex if*

$$\langle x, y\rangle \subset D, \quad \text{for all } x, y \in D.$$

We consider now a map $E : N \to N$.

Definition 3. [7]. *A set $M \subset N$ is said to be E-d-convex if*

$$\langle E(x), E(y)\rangle \subset M, \quad \text{for each } x, y \in M.$$

Theorem 4. [7]. *If a set $M \subset N$ is E-d-convex then $E(M) \subseteq M$.*

Theorem 5. [7]. *If $E(M)$ is d-convex and $E(M) \subseteq M$ then M is E-d-convex.*

2. E-d-MONOTONE FUNCTIONS ON UNDIRECTED TREE NETWORKS

We consider an undirected tree network $N = (V, U)$, a map $E : N \to N$ and two points $x, y \in N$.

We denote

$$\langle x, y\rangle_E = \Big\{ z \in N \mid d\big(E(x), E(y)\big) = d\big(E(x), E(z)\big) + d\big(E(z), E(y)\big) \Big\}.$$

Obviously $E(\langle x, y\rangle_E) \subseteq \langle E(x), E(y)\rangle$. Generally, the converse inequality is not true.

Now, let us define the following order relation on $\langle x, y\rangle_E$. For $z_1, z_2 \in \langle x, y\rangle_E$ with

$$d\big(E(x), E(z_1)\big) = \alpha_1 d\big(E(x), E(y)\big)$$

and

$$d\big(E(x), E(z_2)\big) = \alpha_2 d\big(E(x), E(y)\big),$$

$0 \le \alpha_1 \le 1$, $0 \le \alpha_2 \le 1$, we say that $z_1 \le_E z_2$ if $\alpha_1 \le \alpha_2$. If two points $z_1, z_2 \in \langle x, y\rangle_E$ satisfy the equality $E(z_1) = E(z_2)$ then we shall write $z_1 =_E z_2$.

Obviously $x \le_E y$ and every $z \in \langle x, y\rangle_E$ satisfies $x \le_E z \le_E y$.

We consider now the function $f : N \to \mathbb{R}$ and the points $x, y \in N$.

Definition 6. (1) *The function $f : N \to \mathbb{R}$ is said to be E-d-increasing between x and y if for every $z_1, z_2 \in \langle x, y\rangle_E$ such that $z_1 \le_E z_2$ we have*

$$f\big(E(z_1)\big) \le f\big(E(z_2)\big).$$

(2) *The function $f : N \to \mathbb{R}$ is said to be E-d-decreasing between x and y if for every $z_1, z_2 \in \langle x, y\rangle_E$ such that $z_1 \le_E z_2$ we have*

$$f\big(E(z_1)\big) \ge f\big(E(z_2)\big).$$

(3) *The function $f : N \to \mathbb{R}$ is said to be E-d-constant between x and y if*

$$f\big(E(x)\big) = f\big(E(z)\big), \quad \textit{for every } z \in \langle x, y\rangle_E\,.$$

A function that is either E-d-increasing or E-d-decreasing between x and y is said to be E-d-monotone between x and y. If all the inequalities in Definition 6 are strict then f is called strictly E-d-increasing, strictly E-d-decreasing or strictly E-d-monotone.

Remarks. 1. If a function $f : N \to \mathbb{R}$ is both E-d-increasing and E-d-decreasing between x and y then it is E-d-constant between x and y.

2. If a function $f : N \to \mathbb{R}$ is E-d-increasing between x and y then it is E-d-decreasing between y and x. □

In the following we give an example of E-d-monotone function.

Example 1. We consider a tree network $N=(V,U)$ with $V=\{v_1,v_2,v_3\}$ and $U=\{\,[v_1,v_2]\,,[v_1,v_3]\,\}$ such that $l(\,[v_1,v_2]\,)=1$ and $l(\,[v_1,v_3]\,)\geq 1$. For every edge $[v_i,v_j]\in U$ we consider the corresponding function $\theta_{ij}:[v_i,v_j]\to[0,1]$. For every $z\in[v_i,v_j]$ we denote by z' the point of the edge $[v_1,v_2]$ such that $d(v_1,z')=\theta_{ij}(z)$ and let $l_z=\theta_{ij}(z)$. We define now $E:N\to N$,

$$E(z)=z',\quad \forall z\in N.$$

For the points $x=v_1$ and $y=v_2$ we have

$$\langle x,y\rangle_E=N\quad\text{and}\quad E(\langle x,y\rangle_E)=[v_1,v_2]\,.$$

The function $f:N\to\mathbb{R}$,

$$f(z)=l_z,\quad \forall z\in N$$

is E-d-increasing between x and y. Indeed, if $z_1,z_2\in\langle x,y\rangle_E$ and $z_1\leq_E z_2$ then $l_{z_1}\leq l_{z_2}$ and consequently $f(z_1)\leq f(z_2)$. □

3. FUNCTIONS WITH BOUNDED E-d-VARIATION ON UNDIRECTED TREE NETWORKS

We consider an undirected tree network $N=(V,U)$, a map $E:N\to N$, an E-d-convex set $M\subset N$ and the function $f:M\to\mathbb{R}$. We also consider the points $x,y\in M$.

For a division (σ) of the set $\langle x,y\rangle_E\cap M$ by the points

$$x=_E z_0<_E z_1<_E z_2<_E\ldots<_E z_q=_E y \tag{2}$$

we define the number

$$\bigvee(E;f,\sigma)=\sum_{i=1}^{q}\left|f(E(z_i))-f(E(z_{i-1}))\right|,$$

called the E-d-variation of the function f on the division (σ).

We denote by D the set of all divisions (σ) of the set $\langle x,y\rangle_E\cap M$.

Definition 7. *The number*

$$\bigvee_x^y(E;f)=\sup\left\{\bigvee(E;f,\sigma)\mid\sigma\in D\right\}$$

is called the total E-d-variation of the function f on $\langle x,y\rangle_E\cap M$.

Definition 8. *The function $f:M\to\mathbb{R}$ is said to be with bounded E-d-variation on $\langle x,y\rangle_E\cap M$ if*

$$\bigvee_x^y(E;f)<\infty.$$

Theorem 9. *If the function $f:M\to\mathbb{R}$ is E-d-monotone between x and y then it is with bounded E-d-variation on $\langle x,y\rangle_E\cap M$.*

Proof. If the function $f : M \to \mathbb{R}$ is E-d-increasing between x and y then for every division (2)

$$\begin{aligned} \bigvee(E; f, \sigma) &= \sum_{i=1}^{q} \Big| f\big(E(z_i)\big) - f\big(E(z_{i-1})\big) \Big| \\ &= \sum_{i=1}^{q} \Big[f\big(E(z_i)\big) - f\big(E(z_{i-1})\big) \Big] \\ &= f\big(E(y)\big) - f\big(E(x)\big), \end{aligned}$$

and therefore

$$\bigvee_{x}^{y} (E; f) = f\big(E(y)\big) - f\big(E(x)\big) < \infty.$$

If the function is E-d-decreasing between x and y then the proof is analogously. □

Corollary 10. *If the function $f : M \to \mathbb{R}$ is E-d-constant between x and y then it is with bounded E-d-variation on $\langle x, y \rangle_E \cap M$ and the total E-d-variation of f on $\langle x, y \rangle_E \cap M$ is zero.*

We denote by E_1 the restriction of the map $E : N \to \mathbb{R}$ to the E-d-convex set M. Since the set M is E-d-convex, $E(M) \subseteq M$.

Theorem 11. *If the function $f : M \to \mathbb{R}$ is with bounded E-d-variation on $\langle x, y \rangle_E \cap M$ then the function $f \circ E_1 : M \to \mathbb{R}$ is bounded on $\langle x, y \rangle_E \cap M$.*

Proof. For the particular division $\sigma = (x <_E z <_E y)$ of $\langle x, y \rangle_E \cap M$ we have

$$\bigvee(E; f, \sigma) = \Big| f\big(E(z)\big) - f\big(E(x)\big) \Big| + \Big| f\big(E(y)\big) - f\big(E(z)\big) \Big| \le \bigvee_{x}^{y} (E; f)$$

and

$$\Big| f\big(E(z)\big) \Big| \le \Big| f\big(E(z)\big) - f\big(E(x)\big) \Big| + \Big| f\big(E(x)\big) \Big| \le \bigvee_{x}^{y} (E; f) + \Big| f\big(E(x)\big) \Big|.$$

Consequently the function $f \circ E_1 : M \to \mathbb{R}$ is bounded on $\langle x, y \rangle_E \cap M$. □

The following two theorems are immediately implied.

Theorem 12. *If the functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are with bounded E-d-variation on $\langle x, y \rangle_E \cap M$ then the functions $f + g$, $f - g$, and fg are with bounded E-d-variation on $\langle x, y \rangle_E \cap M$.*

Theorem 13. *If the functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are with bounded E-d-variation on $\langle x, y \rangle_E \cap M$ and there is a number $\eta > 0$ such that*

$$g(E(z)) \ge \eta, \quad \textit{for every } z \in \langle x, y \rangle_E \cap M,$$

then the function f/g is with bounded E-d-variation on $\langle x, y \rangle_E \cap M$.

DEFINITION 14. *The function $f : M \to \mathbb{R}$ satisfies the E-Lipschitz condition on $\langle x, y\rangle_E \cap M$ if there is a number $k > 0$ such that for any pair of points $z_1, z_2 \in \langle x, y\rangle_E \cap M$ it is satisfied the relation:*

$$|f(E(z_1)) - f(E(z_2))| \leq kd\big(E(z_1), E(z_2)\big). \tag{3}$$

THEOREM 15. *If the function $f : M \to \mathbb{R}$ satisfies the E-Lipschitz condition on $\langle x, y\rangle_E \cap M$ then it is with bounded E-d-variation on $\langle x, y\rangle_E \cap M$.*

Proof. Indeed, if the function $f : M \to \mathbb{R}$ satisfies the E-Lipschitz condition on $\langle x, y\rangle_E \cap M$ then there is a number $k > 0$ such that for any pair of points $z_1, z_2 \in \langle x, y\rangle_E \cap M$ is satisfied the relation (3). Consequently we have:

$$\begin{aligned}
\bigvee(E; f, \sigma) &= \sum_{i=1}^{q} |f(E(z_i)) - f(E(z_{i-1}))| \\
&\leq \sum_{i=1}^{q} kd\big(E(z_i), E(z_{i-1})\big) \\
&\leq kd\big(E(x), E(y)\big) \\
&< \infty
\end{aligned}$$

and hence $\bigvee\limits_x^y (E; f) < \infty$ and f is with bounded E-d-variation on $\langle x, y\rangle_E \cap M$. □

We consider now a point

$$z \in \big(\langle x, y\rangle_E \cap M\big) \backslash \{t \in M \mid t =_E x \text{ or } t =_E y\}.$$

THEOREM 16. *If the function $f : M \to \mathbb{R}$ is with bounded E-d-variation on $\langle x, y\rangle_E \cap M$ then it is with bounded E-d-variation on $\langle x, z\rangle_E \cap M$ and on $\langle z, y\rangle_E \cap M$ and*

$$\bigvee_x^y (E; f) = \bigvee_x^z (E; f) + \bigvee_z^y (E; f). \tag{4}$$

THEOREM 17. *If the function $f : M \to \mathbb{R}$ is with bounded E-d-variation on $\langle x, z\rangle_E \cap M$ and on $\langle z, y\rangle_E \cap M$ then it is with bounded E-d-variation on $\langle x, y\rangle_E \cap M$.*

In the following lines we will establish a Jordan type theorem.

THEOREM 18. *The function $f : \langle x, y\rangle_E \cap M \to \mathbb{R}$ is with bounded E-d-variation on $\langle x, y\rangle_E \cap M$ if and only if there exist two E-d-increasing functions between x and y, $g : \langle x, y\rangle_E \cap M \to \mathbb{R}$ and $h : \langle x, y\rangle_E \cap M \to \mathbb{R}$ such that $f = g - h$.*

Proof. The sufficiency of the condition follows from Theorem 9 and Theorem 12.

For the necessity, let us define the function

$$g : \langle x, y\rangle_E \cap M \to \mathbb{R}, \quad g(z) = \begin{cases} \bigvee\limits_x^z (E; f), & \text{if } x <_E z \leq_E y \\ 0, & \text{if } x =_E z. \end{cases}$$

From Theorem 16 follows that the function g is E-d-increasing between x and y.

We define now the function

$$h : \langle x, y\rangle_E \cap M \to \mathbb{R}, \quad h(z) = g(z) - f(z).$$

This function is E-d-increasing between x and y. Indeed, if we consider the points z', $z'' \in \langle x, y\rangle_E \cap M$, such that $z' <_E z''$, we have

$$\begin{aligned} h(z'') &= g(z'') - f(z'') = g(z') + \bigvee_{z'}^{z''} (E; f) - f(z''), \\ h(z'') - h(z') &= \bigvee_{z'}^{z''} (E; f) - \left[f(z'') - f(z')\right]. \end{aligned}$$

But

$$f(z'') - f(z') \leq \bigvee_{z'}^{z''} (E; f).$$

Consequently $h(z'') - h(z') \geq 0$, that is, h is E-d-increasing between x and y. Hence $f = g - h$, where the functions g and h are E-d-increasing between x and y. □

Remark 1. The representation of a function with bounded E-d-variation as a difference of two E-d-increasing functions is not unique. Indeed, if $f = g - h$ and the functions g and h are E-d-increasing between x and y then we also have

$$f = g + c - (h + c),$$

c being constant on $\langle x, y\rangle_E \cap M$. The functions $g+c$ and $h+c$ are E-d-increasing between x and y too. □

Remark 2. In [6] we already defined and studied another class of functions with bounded variation on undirected networks. □

REFERENCES

[1] Cristescu, G., *Functions of bounded E-variation*, Séminaire de la Théorie de la Meilleure Approximation, Convexité et Optimisation, Cluj-Napoca, October 26–29, pp. 73–85, 2000.

[2] Dearing, P. M. and Francis, R. L., *A minimax location problem on a network*, Transportation Science, **8**, pp. 333–343, 1974.

[3] Dearing, P. M., Francis, R. L. and Lowe, T. J., *Convex location problems on tree networks*, Oper. Res. **24**, pp. 628–634, 1976.

[4] Iacob, M. E., *Convexity, approximation and optimization on networks*, PhD thesis, Babeş-Bolyai University, Cluj-Napoca, 1997.

[5] Labeé, M., *Essay in network location theory*, Cahiers de Centre d'Etudes et Recherche Oper., **27**, 1–2, pp. 7–130, 1985.

[6] Marian, D., *Generalized convex functions and mathematical analysis on networks*, Research on Theory of Allure, Approximation, Convexity and Optimization, Cluj-Napoca, pp. 183–206, 1999.

[7] Marian, D., *An axiomatic approach to the theory of E-convex functions*, Proceedings of the "Tiberiu Popoviciu" Itinerant Seminar of Functional Equations, Approximation and Convexity, Cluj-Napoca, May 22–26, pp. 97–106, 2001.

[8] Popoviciu, E., *Mean Value Theorems in Mathematical Analysis and their Connection to the Interpolation Theory* , Ed. Dacia, Cluj-Napoca, Romania, 1972 (in Romanian).

[9] Youness, E. A., *E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory Appl., **102**, no. 2, pp. 439–450, 1999.

Received by the editors: January 16, 2001.