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APPROXIMATION OF DERIVATIVES BY NONLINEAR OPERATORS

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Abstract. There are obtained two theorems on simultaneous approximation, by using generalized convex operators.

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1. INTRODUCTION

In [6] Tiberiu Popoviciu obtained that Bernstein operators preserve convexity of higher orders. On the other hand the sequence of Bernstein operators has the property of the uniform approximation of the derivatives of higher order. This is a fact more general. Sendov and Popov obtained in [8] that, roughly speaking, if a sequence of linear positive operators that preserve convexity of higher orders has the property of the uniform approximation of continuous functions then it has also the propriety of the uniform approximation of derivatives of higher orders on any compact subinterval strictly contained in the interval of definition of functions.

In this paper we shall obtain two theorems concerning the uniform approximation of derivatives of higher orders by using sequences of nonlinear operators having the propriety of preservation of some type of generalized convexity of higher orders. As regard to [8] our scheme of the proof is simplified, but it requires a supplementary order of derivability.

2. CONVEX OPERATORS FOR APPROXIMATION OF VECTOR-VALUED FUNCTIONS

Let $[a,b]$ be an interval of the real axis and let F be an Euclidean space with the scalar product $\langle , \rangle$ and the corresponding norm $\|\cdot\|$. Denote respectively by $\mathcal{F}([a,b],F)$ the space of functions defined on $[a,b]$ and with values in F, by $C([a,b],F)$ the subspace of continuous funtions, endowed with the Chebysev norm $\|\cdot\|_{[a,b]}$ and for the integer $m\geq 1$ denote by $C^m([a,b],F)$ the subspace of m times continuously derivable functions. In the the case $F=\mathbb{R}$ we omit to write F. If $x_0,x_1,\ldots,x_{m+1},\ m\geq -1$ are distinct points of $[a,b]$ then, for a function $f:[a,b]\to F$ denote by $[f;x_0,x_1,\ldots,x_{m+1}]$ the divided difference

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of function f on the points x_i, $0 \leq i \leq m+1$. We introduce the following definition:

DEFINITION 1. *A function* $f : [a,b] \to F$ *is c-nonconcave of order* $m \geq -1$, *if for any choice of two sequence of distinct points* $x_0, x_1, \ldots, x_{m+1}$ *and* $y_0, y_1, \ldots, y_{m+1}$ *of* $[a,b]$ *we have:*

$$\big\langle [f; x_0, x_1, \ldots, x_{m+1}], [f; y_0, y_1, \ldots, y_{m+1}] \big\rangle \geq 0. \tag{1}$$

In a similar mode, by replacing in (1) *the inequality "$\geq$" by "$>$", or "$=$" one can define the functions that are c-convex, respectively c-polynomial of order* m. *Denote by* $K_m([a,b],F)$ *the space of functions that are c-nonconcave of order* m.

REMARK. In the case $F = \mathbb{R}$ a function is c-nonconcave of order $m \geq -1$ if and only if it is either usual nonconcave of order m or it is usual nonconvex of order m (see [5]). $\square$

LEMMA 2. *If* $a, b, v \in F$, $\|a\| = \|b\| = \|v\| = 1$, $\langle a, b\rangle \leq 0$ *then* $\max\{\langle a, v\rangle, \langle b, v\rangle\} \geq -\sqrt{2}/2$.

Proof. Let $p := \dim F$. If $p = 1$ Lemma 2 is obvious. Let $p \geq 2$. We have $a \neq b$. First consider the case $a \neq -b$. Let $\{\epsilon_1, \ldots, \epsilon_p\}$ be an orthonormal basis of the space F such that $\epsilon_1 = (a+b)/\|a+b\|$ and $\epsilon_2 = (b-a)/\|b-a\|$. Represent $v = \lambda_1 \cdot \epsilon_1 + \ldots + \lambda_p \cdot \epsilon_p$, where $(\lambda_1)^2 + \ldots + (\lambda_p)^2 = 1$. We have $\langle v, a\rangle = \lambda_1(1+\langle a,b\rangle)/\|a+b\| + \lambda_2(\langle b,a\rangle - 1)/\|b-a\|$; $\langle v, b\rangle = \lambda_1(\langle a,b\rangle+1)/\|a+b\| + \lambda_2(1-\langle a,b\rangle)/\|b-a\|$. Hence $\max\{\langle a,v\rangle, \langle b,v\rangle\} = \lambda_1(1+\langle a,b\rangle)/\|a+b\| + |\lambda_2|(1-\langle a,b\rangle)/\|b-a\|$. By considering a and b fixed one obtains the minimum value of $\max\{\langle a,v\rangle, \langle b,v\rangle\}$ in the case $\lambda_1 = -1$ and $\lambda_2 = \ldots = \lambda_p = 0$ and it is equal to $-(1+\langle a,b\rangle)/\|a+b\| = -(1/\sqrt{2})\sqrt{1+\langle a,b\rangle} \geq -\sqrt{2}/2$. The case $a = -b$ is immediate. $\square$

THEOREM 3. *If the sequence of functions* $(f_n)_n$, $f_n \in C^1([a,b],F)$ *is uniformly convergent to the function* $f \in C^1([a,b],F)$ *on* $[a,b]$ *and if* $f_n' \in K_0([a,b],F)$, $n \in \mathbb{N}$, *then, for any subinterval* $[c,d] \subset (a,b)$, *the sequence* $(f_n')_n$ *is uniformly convergent on* $[c,d]$ *to the function* f'.

Proof. Consider ad absurdum that there is a number $\lambda > 0$, a sequence $(x_k)_k$ of points $x_k \in [c,d]$ and a subsequence $(n_k)_k$ of indices such that

$$\|f'(x_k) - f_{n_k}'(x_k)\| > \lambda, \qquad k \in \mathbb{N}.$$

There is a number $\delta_1 > 0$ such that

$$\|f'(x) - f'(y)\| < \lambda/4, \qquad \text{if} \quad |x-y| < \delta_1.$$

Put $\delta := \min\{\delta_1, c-a, b-d\}$ and $\rho := \lambda\delta/8$. Afterwards fix k such that

$$\|f - f_{n_k}\|_{[a,b]} < \rho.$$

Define $g := f_{n_k}$, $y := x_k$ and

$$I_1 := \int_{y-\delta}^{y} (g'(t) - g'(y))\,\mathrm{d}t,$$

$$I_2 := \int_{y}^{y+\delta} (g'(t) - g'(y))\,\mathrm{d}t.$$

Here it is used the Riemann integral for functions with values in an Euclidean space.

Since $g' \in K_0([a,b], f)$ it follows for any points $a \le t_1 < y < t_2 \le b$:

$$\langle g'(t_1) - g'(y), g'(t_2) - g'(y)\rangle \le 0.$$

By approximating I_1 and I_2 by Riemann sums we obtain

$$\langle I_1, I_2\rangle \le 0.$$

First consider the case $I_1 \neq 0$ and $I_2 \neq 0$. Set $\alpha := I_1/\|I_1\|$, $\beta := I_2/\|I_2\|$ and $v := u/\|u\|$, where $u := \delta(g'(y) - f'(y))$. From Lemma 2 it follows $\max\{\langle\alpha, v\rangle, \langle\beta, v\rangle\} \ge -\sqrt{2}/2$. Suppose, for a choice, that $\langle\beta, v\rangle \ge -\sqrt{2}/2$. We have

$$\begin{aligned}
&\|g(y+\delta) - f(y+\delta)\| \\
&= \left\| g(y) + \int_y^{y+\delta} g'(t)\,\mathrm{d}t \;-\; f(y) \;-\; \int_y^{y+\delta} f'(t)\,\mathrm{d}t \right\| \\
&\ge \left\| \int_y^{y+\delta} (g'(t) - f'(t))\,\mathrm{d}t \right\| \;-\; \rho \\
&\ge \left\| \int_y^{y+\delta} (g'(t) - f'(y))\,\mathrm{d}t \right\| \;-\; \left\| \int_y^{y+\delta} (f'(y) - f'(t))\,\mathrm{d}t \right\| \;-\; \rho \\
&\ge \left\| \int_y^{y+\delta} (g'(t) - f'(y))\,\mathrm{d}t \right\| \;-\; 3\rho \\
&= \;\|I_2 + u\| \;-\; 3\rho \\
&\ge \left(\|I_2\|^2 + \|u\|^2 - \sqrt{2}\|I_2\| \cdot \|u\| \right)^{1/2} \;-\; 3\rho \\
&\ge \;\|u\|/\sqrt{2} \;-\; 3\rho \\
&> \lambda\delta/\sqrt{2} \;-\; 3\rho \\
&> \rho.
\end{aligned}$$

One obtains a contradiction. In the case $I_2 = 0$ one obtains as above that $\|g(y+\delta) - f(y+\delta)\| \ge \|u\| - 3\rho > \rho$. The case $I_1 = 0$ is similar. Theorem is proved. □

Remark. In the case $F = \mathbb{R}$ the result in Theorem 3 is given in [8]. □

Lemma 4. *Let $[a,b] \subset \mathbb{R}$ and $f : [a,b] \to F$. For any $x_i, \in \mathbb{R}$, $1 \le i \le n$, and any $t \in \mathbb{R}$ such that x_i and $x_i + t$, $1 \le i \le n$, are $2n$ distinct points of* *the interval* $[a,b]$, *by denoting* $f_t(x) := (f(x+t) - f(x))/t$, $x \in [\max\{a, a - t\}, \min\{b, b-t\}]$, *we have*

$$(2)\qquad [f_t; x_1, \ldots, x_n] = \sum_{k=1}^{n} [f; x_1 + t, \ldots, x_{k-1} + t, x_k + t, x_k, x_{k+1}, \ldots, x_n].$$

Proof. For $p \geq 1$, $u \neq 0$, y_i, $1 \leq i \leq p$, such that $y_1 \neq y_i$, $y_1 \neq y_i - u$, $2 \leq i \leq p$, denote

$$\Theta_u^p(y_1, \ldots, y_p) := \sum_{j=1}^{p} \prod_{i=2}^{j} (y_1 - y_i)^{-1} \cdot \prod_{i=j}^{p} (y_1 + u - y_i)^{-1}.$$

(For $j = 1$ take $\prod_{i=2}^{j} = 1$). Using the relation

$$\Theta_u^{p+1}(y_1, \ldots, y_{p+1}) = (y_1 + u - y_{p+1})^{-1} \Big[\Theta_u^p(y_1, \ldots, y_p) + \prod_{i=2}^{p+1} (y_1 - y_i)^{-1}\Big],$$

one can proved by induction with regard to p that

$$\Theta_u^p(y_1, \ldots, y_p) = \Big(u \cdot \prod_{i=2}^{p} (y_1 - y_i)\Big)^{-1}.$$

We have

$$\begin{aligned}
[f_t; x_1, \ldots, x_n] &= \sum_{k=1}^{n} \frac{f(x_k + t) - f(x_k)}{t} \prod_{1 \leq i \leq n, i \neq k} (x_k - x_i)^{-1} \\
&= \sum_{k=1}^{n} f(x_k + t) \cdot \Theta_t^{n-k+1}(x_k, \ldots, x_n) \cdot \prod_{i=1}^{k-1} (x_k - x_i)^{-1} \\
&\quad + \sum_{k=1}^{n} f(x_k) \cdot \Theta_{-t}^{k}(x_k + t, \ldots, x_1 + t) \cdot \prod_{i=k+1}^{n} (x_k - x_i)^{-1} \\
&= \sum_{k=1}^{n} [f; x_1 + t, \ldots, x_{k-1} + t, x_k + t, x_k, x_{k+1}, \ldots, x_n]. \qquad \square
\end{aligned}$$

THEOREM 5. *If* $f \in C^k([a,b], F) \cap K_m([a,b], F)$, $k \geq 1, m - k + 1 \geq 0$, *then* $f^{(k)} \in K_{m-k}([a,b], F)$.

Proof. Using Lemma 4 it is easy to obtain that if

$$f \in C^1([a,b], f) \cap K_{n-1}([a,b], F), \quad n \geq 1,$$

then it follows $f' \in \cap K_{n-2}([a,b], F)$. Afterwards the theorem results by induction. □

Now consider the following definition.

DEFINITION 6. *An operator* $L : V \to \mathcal{F}([a,b], F)$, $V \subset \mathcal{F}([a,b], F)$ *is said to be* k*-convex,* $k \geq -1$, *if*

$$(3)\qquad L(f) \in K_k([a,b], F), \quad \textit{for any } f \in V \cap K_k([a,b], F)$$

and

(4) $L(f) - L(g) \in K_k\big([a,b],F\big)$, *for any* $f,g \in V,\ f-g \in K_k\big([a,b],F\big)$.

The main result of this section is the following one.

THEOREM 7. *Let* $(L_n)_n,\ L_n : C^{k+1}\big([a,b],F\big) \to C^{k+1}\big([a,b],F\big),\ k \geq 1$, *be a sequence of* j*-convex operators, for* $1 \leq j \leq k$. *If we have*

(5) $$\lim_{n\to\infty} \|L_n(f) - f\|_{[a,b]} = 0, \quad \textit{for all } f \in C^{k+1}\big([a,b],F\big),$$

then for any $f \in C^{k+1}\big([a,b],F\big)$, *for any subinterval* $[c,d] \subset (a,b)$ *and for any* $j,\ 1 \leq j \leq k$, *one has*

(6) $$\lim_{n\to\infty} \big\|(L_n(f))^{(j)} - f^{(j)}\big\|_{[c,d]} = 0.$$

Proof. Fix the interval $[c,d]$ and let the subintervals $[c_j,d_j]$, $1 \leq j \leq k$ such that $[c_0,d_0] = [a,b]$, $(c_j,d_j) \supset [c_{j+1},d_{j+1}]$ and $[c_k,d_k] = [c,d]$. First note that for any function $g \in C^m\big([a,b],F\big)$, $m \geq 0$ and for any points $y_0 < \ldots < y_m$ of $[a,b]$ we have $(m!)\|[g;y_0,\ldots,y_m]\| \leq \|g^{(m)}\|_{[a,b]}$. This one is a consequence of the Peano's formula:

$$[g;y_o,\ldots,y_m] = \int_{y_0}^{y_m} \phi(t)\cdot g^{(m)}(t)\,\mathrm{d}t,$$

where $\phi : [y_0,y_m] \to \mathbb{R}$ is a continuous positive function independent of g.

Fix now $f \in C^{k+1}\big([a,b],F\big)$. Denote $\rho := \max\{|a|,|b|\}$ We can choose by induction the numbers $\lambda_j > 0$, $2 \leq j \leq k+1$, such that:

$$\begin{aligned}(\lambda_j)^2 \ \geq 2\lambda_j\Big(\sum_{i=j+1}^{k+1} \tbinom{i}{j}\rho^{i-j}\cdot\lambda_i + \ (j!)^{-1}\|f^{(j)}\|_{[a,b]}\Big)\\ + \Big(\sum_{i=j+1}^{k+1} \tbinom{i}{j}\rho^{i-j}\cdot\lambda_i + \ (j!)^{-1}\|f^{(j)}\|_{[a,b]}\Big)^2, \qquad 2 \leq j \leq k+1,\end{aligned}$$

(for $j = k+1$ take $\sum_{i=j+1}^{k+1} = 0$). Let $v \in F$ with $\|v\| = 1$, and consider the function:

$$h(x) := f(x) + \Big(\sum_{j=2}^{k+1} \lambda_j \cdot x^j\Big)v, \quad x \in [a,b].$$

We have $h, h-f \in K_j([a,b],F)$, $(1 \leq j \leq k)$. Indeed, let $2 \leq j \leq k+1$ and two sets of distinct points of I: $x_0,\ldots,x_j$ and $y_0,\ldots,y_j$. Using the inequalities above one obtains:

$$\begin{aligned}&\Big\langle [h;x_0,\ldots,x_j],\ [h;y_0,\ldots,y_j]\Big\rangle =\\ &= \Big\langle \Big(\sum_{i=j}^{k+1}\tbinom{i}{j}\lambda_i\xi_i^{i-j}\Big)v + [f;x_0,\ldots,x_j],\ \Big(\sum_{i=j}^{k+1}\tbinom{i}{j}\lambda_i\eta_i^{i-j}\Big)v + [f;y_0,\ldots,y_j]\Big\rangle\\ &\geq 0,\end{aligned}$$

where $\xi_i, \eta_i \in [a,b]$, for $j \le i \le k+1$. Hence $h \in K_{j-1}([a,b],F)$. In a similar mode we can see that $h - f \in K_{j-1}([a,b],F)$.

From Theorem 5 it follows $(L_n(h))^{(j)}$, $(L_n(h) - L_n(f))^{(j)} \in K_0([a,b],F)$, $1 \le j \le k$, $n \ge 1$, and from Theorem 3 it can deduce by induction, for $1 \le j \le k$:

$$\lim_{n\to\infty} \|(L_n(h))^{(j)} - (h)^{(j)}\|_{[c_j,d_j]} = 0 = \lim_{n\to\infty} \|(L_n(h) - L_n(f))^{(j)} - (h-f)^{(j)}\|_{[c_j,d_j]}.$$

From these limits it follows (6). □

3. CONVEX OPERATORS FOR APPROXIMATION OF REAL-VALUED FUNCTIONS

Recall that for $n \ge 1$, a subset $Z \subset C[a,b]$ is named n-parameter family if for any distinct points $x_i \in [a,b]$, $1 \le i \le n$, and any real numbers y_i, $1 \le i \le n$ there is an unique $\psi \in Z$ such that $\psi(x_i) = y_i$, $1 \le i \le n$. Convexity with regard to a n-parameter family was introduced by Tiberiu Popoviciu in [7] and was extensively studied by L. Tornheim in [9] and E. Popoviciu in [3] (and in others).

Definition 8. [7]. *If $Z \subset C[a,b]$ is a n-parameter family, $n \ge 1$, then a function $f \in C[a,b]$ is named Z-convex if for any points $a \le x_1 < \ldots < x_n < t \le b$, it results $f(t) > \psi(t)$, where $\psi \in Z$ is the unique function such that $\psi(x_i) = y_i$, $1 \le i \le n$.*

We consider the following definition.

Definition 9. *Let $Z \subset C[a,b]$ be a n-parameter family, $n \ge 1$. An operator $L : C[a,b] \to \mathcal{F}[a,b]$ is Z-convex if the following conditions are verified*

(7) *If $f \in C[a,b]$ is Z-convex then $L(f)$ is usual convex of order $n-1$*

(8) *If $f - g$ is Z-convex , $f, g \in C[a,b]$, then $L(f) - L(g)$ is usual convex of order $n-1$.*

The main result of this section is the following.

Theorem 10. *Let $k \ge 1$ and for each $2 \le j \le k+1$ let $Z_j \subset C^{k+1}[a,b]$ be a j-parameter family. Suppose that there are the numbers $M_j > 0$ such that*

$$\|\varphi^{(j)}\|_{[a,b]} \le M_j, \quad \text{for any } \varphi \in Z_j,\ 2 \le j \le k+1. \tag{9}$$

If $(L_n)_n$ is a sequence of operators $L_n : C^{k+1}[a,b] \to C^{k+1}[a,b]$ such that

$$L_n \text{ is } Z_j\text{-convex for any } n \ge 1 \text{ and } 2 \le j \le k+1, \tag{10}$$

$$\lim_{n\to\infty} \|L_n(f) - f\|_{[a,b]} = 0, \text{ for any } f \in C^{k+1}[a,b], \tag{11}$$

then for any $f \in C^{k+1}[a,b]$, any subinterval $[c,d] \subset (a,b)$ and any j, $1 \le j \le k$, one has

$$\lim_{n\to\infty} \|(L_n(f))^{(j)} - f^{(j)}\|_{[c,d]} = 0. \tag{12}$$

Proof. First note that if $g \in C^j[a,b]$, $j \geq 2$ and $g^{(j)}(x) > M_j$, $x \in [a,b]$, then g is Z_j-convex. Indeed, let $a \leq y_1 < \ldots < y_j < t \leq b$ and $\varphi \in Z_j$ the unique function such that $\varphi(x_i) = g(x_i)$, $1 \leq i \leq j$. Since the function $g - \varphi$ is usual $j-1$ convex it follows $g(t) > \varphi(t)$.

Now fix $f \in C^{k+1}[a,b]$ and $[c,d] \subset (a,b)$. Denote $\rho := \max\{|a|,|b|\}$. We can choose by induction the numbers λ_j, $2 \leq j \leq k+1$, such that

$$(j!)\lambda_j > \sum_{i=j+1}^{k+1} (i!)\rho^{i-j} \cdot \lambda_i \, + \, \|f^{(j)}\|_{[a,b]} \, + \, M_j.$$

(For $j = k+1$ take $\sum_{i=j+1}^{k+1} = 0$). Define the function h by

$$h(x) := f(x) + \sum_{j=2}^{k+1} \lambda_j \cdot x^j, \quad x \in [a,b].$$

Then h and $h - f$ are Z_j-convex for $2 \leq j \leq k+1$ and consequently $L_n(h)$ and $L_n(h) - L_n(f)$, $n \geq 1$, are usual convex of order $j-1$, for the same j.

Consider the intervals $[c_j, d_j]$, $0 \leq j \leq k$, such that $[c_0, d_0] = [a,b]$, $[c_{j+1}, d_{j+1}] \subset (c_j, d_j)$, $[c_k, d_k] = [c,d]$. Then by using the result in Theorem 3 in the case $F = \mathbb{R}$, it follows by induction that

$$\lim_{n\to\infty} \|(L_n(h))^{(j)} - h^{(j)}\|_{[c_j,d_j]} = 0 = \lim_{n\to\infty} \|(L_n(h) - L_n(f))^{(j)} - (h-f)^{(j)}\|_{[c_j,d_j]},$$

$1 \leq j \leq k$, and consequently (12) is true. □

REFERENCES

[1] Altomare F. and Campiti, M., *Korovkin-type approximation theory and its applications*, W. de Gruyter Series Studies in Mathematics, **17**, Walter de Gruyter & Co., Berlin, 1994.

[2] Korovkin P. P., *Linear operators and the theory of approximation*, Fitmatgiz, Moscow, 1959 (in Russian).

[3] Moldovan (Popoviciu) E., *Sur une generalization des fonctions convexes*, Mathematica (Cluj), **1(24)**, pp. 49–80, 1959.

[4] Nemeth A. B., *Korovkin's theorem for nonlinear 3-parameter families*, Mathematica (Cluj), **11(34)**, no. 1, pp. 135–136, 1969.

[5] Păltănea R., *Approximation operators and their connection to some particular allures*, PhD Thesis, Babeş-Bolyai Univ., Cluj-Napoca, 1992 (in Romanian).

[6] Popoviciu T., *About the best approximation of functions by polynomials*, Mathematical Monographs, Sec. Mat. Univ. Cluj, **III**, 1937.

[7] Popoviciu T., *Les fonctions convexes*, Herman & Cie, Paris, 1945.

[8] Sendov B. and Popov V., *The convergence of derivatives of linear positive operators*, C. R. Acad. Bulgare Sci., **22**, pp. 507–509, 1969 (in Russian).

[9] Tornheim L., *On n-parameter families of functions and associated convex functions*, Trans. Amer. Math. Soc., **69**, pp. 457–467, 1950.

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