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COMPATIBILITY OF SOME SYSTEMS OF INEQUALITIES

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Abstract. In this paper, necessary and sufficient conditions for the compatibility of some systems of quasi-convex, or convex inequalities are established. Finally a new proof for a theorem of Shioji and Takahashi (1988) is given.

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1. INTRODUCTION

Ky Fan studied in [4] the existence of solutions for some systems of convex inequalities involving lower semicontinuous functions defined on a compact convex set in a topological vector space (all the topological vector spaces considered in this paper are real and Hausdorff). Particularly, he proved the following theorem.

THEOREM A. *Let C be a nonempty compact convex subset of a topological vector space and let $\mathcal{F}$ be a family of real-valued lower semicontinuous convex functions defined on C. Then the following assertions are equivalent:*

(i) *The system of convex inequalities*

$$f(x) \leq 0, \quad f \in \mathcal{F}, \tag{1}$$

is compatible on C, i.e., there exists $x \in C$ satisfying (1)*.*

(ii) *For any n nonnegative numbers α_i with $\sum_{i=1}^{n} \alpha_i = 1$ and for any $f_1, f_2, \ldots, f_n \in \mathcal{F}$, there exists $x \in C$ such that*

$$\sum_{i=1}^{n} \alpha_i f_i(x) \leq 0.$$

For closed results and extensions of Fan's theorem see [5], [7], [8], [10] and [11]. In Section 2 we study the compatibility of some systems of inequalities (1) in the case when all functions $f \in \mathcal{F}$ are quasi-convex (Theorems 3 and 5), respectively convex (Theorems 2 and 6).

Shioji and Takahashi in [10, Th. 1] have established a Fan type theorem in the case when some function of two variables associated to the system of inequalities (1) is convex-like in one of the variables. This theorem receives a new proof in Section 3.

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For any positive integer n we denote by S_n the set

$$S_n = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n, \alpha_1 \geq 0, \alpha_2 \geq 0, \ldots, \alpha_n \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

The standard abbreviations $\operatorname{conv} A$, $\operatorname{cl} A$, $\operatorname{card} A$ are used to define the convex hull, closure and cardinality of a set A, respectively.

2. SYSTEMS OF QUASI-CONVEX INEQUALITIES

We recall that a real-valued function f defined on a convex set C is said to be *quasi-convex* if for every real number α, the set $\{x \in C : f(x) \leq \alpha\}$ is convex.

In proving Theorem 2 we shall need the following lemma which is an analogous result of a classical Fan's section theorem.

LEMMA 1. [6, Th. 2.2]. *Let C be a nonempty compact convex subset of a locally convex topological vector space X and K a nonempty closed convex subset of a topological vector space Y. Let A be a subset of $C \times K$ having the following properties:*

(a) *A is closed;*
(b) *for any $y \in K$, $\{x \in C : (x, y) \in A\}$ is nonempty and convex;*
(c) *for any $x \in C$, $\{y \in K : (x, y) \notin A\}$ is convex (possibly empty).*

Then there exists $x_0 \in C$ such that $\{x_0\} \times K \subset A$.

THEOREM 2. *Let C be a nonempty compact convex subset of a locally convex topological vector space X and let $\mathcal{F}$ be a family of continuous quasi-convex functions $f : C \to \mathbb{R}$, satisfying the condition*

(2) *any convex combination of functions in $\mathcal{F}$ is quasi-convex.*

Then the following assertions are equivalent:

(i) *The system of inequalities (1) is compatible on C.*
(ii) *For each integer n, $1 \leq n \leq \operatorname{card} \mathcal{F}$, for each $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S_n$ and any $f_1, f_2, \ldots, f_n \in \mathcal{F}$ there exists $x \in C$ such that $\sum_{i=1}^{n} \alpha_i f_i(x) \leq 0$.*

Proof. It is clear that (i) implies (ii). In order to prove the reverse implication we shall apply Lemma 1 taking in the posture of Y, the vector space of all continuous functions $f : C \to \mathbb{R}$, endowed with the uniform norm $\|f\| = \max\{|f(x)| : x \in C\}$. Also we take $K = \operatorname{cl}(\operatorname{conv} \mathcal{F})$ (the closure being taken with respect to the uniform topology), $A = \{(x, f) \in C \times K : f(x) \leq 0\}$ and we show that conditions (a), (b), (c) in Lemma 1 are satisfied.

(a) Let $((x_i, f_i))_{i \in I}$ be a net in A converging to (x, f). It follows that $f_i(x) \leq 0$ for each $i \in I$ and $x_i \xrightarrow{X} x$, $f_i \xrightarrow{Y} f$. Let us take an arbitrary $\epsilon > 0$. By the continuity of the function f, there is $i_1 \in I$ such that $\|f - f_i\| < \frac{\epsilon}{2}$, for all $i \in I$, $i > i_1$, and by $f_i \xrightarrow{Y} f$ there is $i_2 \in I$ such that $\|f - f_i\| < \frac{\epsilon}{2}$, for all

$i \in I$, $i > i_2$. Then for every $i \in I$ satisfying $i > i_1$, $i > i_2$ we have

$$\begin{aligned} f(x) &= (f(x) - f(x_i)) + (f(x_i) - f_i(x_i)) + f_i(x_i) \\ &\le f(x) - f(x_i) + \|f - f_i\| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

(b) Let $f \in K = \mathrm{cl}(\mathrm{conv}\,\mathcal{F})$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $\mathrm{conv}\,\mathcal{F}$ uniformly converging to f (such a sequence there exists since Y is a normed space). By (ii) it follows that for each $n \in \mathbb{N}$ there exists $x_n \in C$ such that $f_n(x_n) \le 0$. Since C is compact the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to an $x \in C$. Therefore $((x_{n_k}, f_{n_k}))_{k\in\mathbb{N}}$ is a sequence in A convergent to (x, f) and A being closed, it follows that $f(x) \le 0$. Hence the set $\{x \in C : (x, f) \in A\}$ is nonempty.

According to (2), all the functions in $\mathrm{conv}\,\mathcal{F}$ are quasi-convex. It is easily checked that the quasi-convexity is conserved by the pointwise convergence, hence by the uniform convergence too.

(c) For every $x \in C$, the set $\{f \in K; f(x) > 0\}$ is obviously convex.

So all the conditions of Lemma 1 are satisfied. Therefore there exists $x_0 \in C$ such that $\{x_0\} \times \mathrm{cl}(\mathrm{conv}\,\mathcal{F}) \subset A$. Particularly, for every $f \in \mathcal{F}$ we have $(x_0, f) \in A$, that is, $f(x_0) \le 0$. □

In [10, Th. 2], Shioji and Takahashi extend Fan's theorem to families of lower semicontinuous convex functions with values in $(-\infty, \infty]$. More exactly they have established

THEOREM 3. *Let C be a nonempty compact convex subset of a topological vector space X and let $\mathcal{F}$ be a family of lower semicontinuous convex functions $f : C \to (-\infty, \infty]$. Then the assertions* (i) *and* (ii) *in Theorem* 2 *are equivalent.*

It should be mentioned that in the case when the topological vector space X is locally convex, Theorem 3 can be derived from Theorem 2. Indeed let C be a nonempty compact convex subset of a locally convex space and let $\mathcal{F}$ be a family of lower semicontinuous convex functions $f : C \to (-\infty, \infty]$. It is clear that (i) implies (ii). In order to prove the reverse implication, for each $f \in \mathcal{F}$ let $\mathcal{A}_f$ be the set of all continuous affine functions $g : C \to \mathbb{R}$ satisfying $g(x) \le f(x)$, for all $x \in C$, and denote by $\mathcal{G} = \cup\{\mathcal{A}_f : f \in \mathcal{F}\}$.

It is known (see [2, p. 99] or [9, p. 30]) that for a semicontinuous convex function $f : C \to (-\infty, \infty]$ the following equality holds

$$f(x) = \sup\{g(x) : g \in \mathcal{A}_f\}. \tag{3}$$

We show that the system

$$g(x) \le 0, \quad g \in \mathcal{G},$$

is compatible on C. Obviously, the family $\mathcal{G}$ satisfies condition (2) in Theorem 2. Let n be a positive integer, $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S_n$ and $g_1, g_2, \ldots, g_n \in \mathcal{G}$.

If $g_1, g_2, \dots, g_k \in \mathcal{A}_{f_1}$, $g_{k+1}, g_{k+2}, \dots, g_l \in \mathcal{A}_{f_2}$, $\dots$, $g_{r+1}, g_{r+2}, \dots, g_n \in \mathcal{A}_{f_m}$, then for every $x \in C$ we have

$$\begin{aligned}&\alpha_1 g_1(x) + \alpha_2 g_2(x) + \dots + \alpha_n g_n(x) \le \\ &\le (\alpha_1 + \alpha_2 + \dots + \alpha_k) f_1(x) + (\alpha_{k+1} + \alpha_{k+2} + \dots + \alpha_l) f_2(x) + \dots \\ &\quad + (\alpha_{r+1} + \alpha_{r+2} + \dots + \alpha_n) f_m(x).\end{aligned}$$

The sum in the right-hand side of the above inequality is a convex combination of the functions $f_1, f_2, \dots, f_m$ hence, according to (ii), it is ≤ 0 for at least one $x \in C$. This shows that $\mathcal{G}$ satisfies (ii). Theorem 2 applied to the family of functions $\mathcal{G}$ puts into evidence an $x_0 \in C$ such that $g(x_0) \le 0$ for each $g \in \mathcal{G}$. By (3) it follows immediately that $f(x_0) \le 0$, for each $f \in \mathcal{F}$.

Remark 1. Observe that if the family of functions $\mathcal{F}$ is finite, having $\operatorname{card} \mathcal{F} = n$, the condition (ii) in each of Theorems A, 2, 3 can be replaced by

(ii′) *For each* $(\alpha_1, \alpha_2, \dots, \alpha_n) \in S_n$ *and any* $f_1, f_2, \dots, f_n \in \mathcal{F}$ *there exists* $x \in C$ *such that* $\sum_{i=1}^{n} \alpha_i f_i(x) \le 0$. □

In Theorem 5 we shall give a set of sufficient conditions for the compatibility of systems of convex inequalities. The proof will be based on Theorem 3 and on the following intersectional result for convex sets (see [1] or [3]).

Lemma 4. *Let C be a compact convex subset of a topological vector space, $\mathcal{A}$ a family of closed convex subsets of C, and k, l two positive integers with $k \le l + 1 \le \operatorname{card} \mathcal{A}$. Suppose that*

(i) $\cup \mathcal{A}' = C$, *for any subfamily $\mathcal{A}'$ of $\mathcal{A}$ with* $\operatorname{card} \mathcal{A}' = k$;

(ii) $\cap \mathcal{A}' \ne \emptyset$, *for any subfamily $\mathcal{A}'$ of $\mathcal{A}$ with* $\operatorname{card} \mathcal{A}' = l$.

Then $\cap \mathcal{A} \ne \emptyset$.

Theorem 5. *Let C be a nonempty compact convex subset of a topological vector space, $\mathcal{F}$ be a family of lower semicontinuous convex functions $f : C \to (-\infty, \infty]$, and k, l two positive integers with $k \le l + 1 \le \operatorname{card} \mathcal{F}$. Suppose that*

(a) *for each k functions, pairwise distinct, $f_1, f_2, \dots, f_k \in \mathcal{F}$ and any $x \in C$ there exists $(\alpha_1, \alpha_2, \dots, \alpha_k) \in S_k$ such that*

$$\sum_{j=1}^{k} \alpha_j f_j(x) \le 0;$$

(b) *for each l functions $f_1, f_2, \dots, f_l \in \mathcal{F}$ and any $(\alpha_1, \alpha_2, \dots, \alpha_l) \in S_l$ there exists $x \in C$ such that*

$$f(x) \le 0, \text{ for all } f \in \mathcal{F}.$$

Proof. Denote by $\mathcal{A}$ the family of all sets $A_i = \{x \in C : f_i(x) \le 0\}$, where $f_i \in \mathcal{F}$. Since the functions $f_i \in \mathcal{F}$ are lower semicontinuous and convex, the corresponding sets A_i are closed in C and convex. The proof of Theorem 5 will be achieved whenever we verify the conditions (i) and (ii) in Lemma 4 for the family $\mathcal{A}$.

If $\mathcal{A}$ does not satisfy the condition (i) then there exists k functions, pairwise distinct, $f_1, f_2, \ldots, f_k$ in $\mathcal{F}$ and x in C such that $f_j(x) > 0$, for all $j \in \{1, 2, \ldots, k\}$. But in this case for any $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in S_k$ we have $\sum_{j=1}^{k} \alpha_j f_j(x) > 0$, which contradicts condition (a).

Now given a subfamily $\{A_1, A_2, \ldots, A_l\}$ of l members of $\mathcal{A}$, i.e., $A_j = \{x \in C : f_j(x) \leq 0\}$, $f_j \in \mathcal{F}$, then condition (b) together with Theorem 3, via Remark 1, yield an $x \in C$ such that $f_j(x) \leq 0$, for all $j \in \{1, 2, \ldots, k\}$, that is, $A_1 \cap A_2 \cap \cdots \cap A_l \neq \emptyset$. □

The following result can be proved by applying the same argument as in the previous proof, using Theorem 2 instead of Theorem 3.

THEOREM 6. *Let C be a nonempty compact convex subset of a locally convex topological vector space, $\mathcal{F}$ a family of continuous quasi-convex functions $f : C \to \mathbb{R}$ satisfying condition* (2) *in Theorem* 2, *and k, l two positive integers with $k \leq l + 1 \leq \operatorname{card} \mathcal{F}$. If conditions* (a) *and* (b) *in Theorem* 5 *hold, then there exists $x \in C$ such that*

$$f(x) \leq 0, \text{ for all } f \in \mathcal{F}.$$

3. THE SHIOJI-TAKAHASHI THEOREM

In [10, Th. 1] Shioji and Takahashi have extended Fan's theorem to functions more general than the convex ones. The goal of this section is to give a new proof of this result, using a minimax theorem.

Before going to this result, we first recollect the following definitions (see [2, p. 161]).

Let A, B be arbitrary sets. A function $F : A \times B \to (-\infty, \infty]$ is said to be:

(i) *concave-like in its first variable*, if for any $x_1, x_2 \in A$ and $0 < \alpha < 1$, there exists $x_0 \in A$ such that

$$\alpha F(x_1, y) + (1 - \alpha) F(x_2, y) \leq F(x_0, y), \text{ for all } y \in B;$$

(ii) *convex-like in its second variable*, if for any $y_1, y_2 \in B$ and $0 < \alpha < 1$, there exists $y_0 \in B$ such that

$$F(x, y_0) \leq \alpha F(x, y_1) + (1 - \alpha) F(x, y_2), \text{ for all } x \in A;$$

(iii) *concave-convex-like*, if it is concave-like in its first variable and convex-like in its second variable.

REMARK 2. It is clear from condition (i) that the following property results

(i′) for every $x_1, x_2, \ldots, x_n \in A$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S_n$, there exists $x_0 \in A$ such that

$$\sum_{i=1}^{n} \alpha_i F(x_i, y) \leq F(x_0, y), \quad \text{for all } y \in B.$$

A similar statement for condition (ii) holds. □

LEMMA 7. *Let A and B be compact topological spaces and let $F : A\times B \to \mathbb{R}$ be an upper-lower semicontinuous concave-convex-like function. Then*

$$\max_{x\in A}\ \min_{y\in B} F(x,y) = \min_{y\in B}\ \max_{x\in A} F(x,y).$$

The above lemma has been formulated in [2, Th. 3.5], in the case when A and B are compact convex sets, each in a topological vector space, but the proof given there holds too in the conditions imposed by us.

The following theorem was obtained by Shioji and Takahashi in [10, Th. 1]. We present another proof relied on Lemma 7.

THEOREM 8. *Let C be a nonempty compact space* (*not necessarily Hausdorff*). *Let $\mathcal{F}$ be a family of lower semicontinuous functions $f : C \to \mathbb{R}$ such that the function $F : \mathcal{F}\times C \to \mathbb{R}$ defined by $F(f,x) = f(x)$, for each $f \in \mathcal{F}$ and $x \in C$, is convex-like in its second variable. Then the assertions* (i) *and* (ii) *in Theorem* 2 *are equivalent.*

Proof. We have only to prove the implication $(ii) \Rightarrow (i)$. The set C being compact and the functions $f \in \mathcal{F}$ being lower semicontinuous, it follows immediately that an infinite system (1) is compatible if and only if every finite subsystem is compatible. So, we may assume that the family $\mathcal{F}$ is finite, namely $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$. Define the function $L : S_n \times C \to \mathbb{R}$ by

$$L(\alpha,x) = \sum_{i=1}^{n} \alpha_i f_i(x), \qquad \text{for each } (\alpha_1,\alpha_2,\dots,\alpha_n) \in S_n \text{ and } x \in C.$$

Clearly L is linear, hence continuous in its first variable. On the other side, from hypothesis it follows that L is lower semicontinuous convex-like in its second variable.

Our assumption (ii) can be written as a minimax inequality, namely

$$\max_{\alpha\in S_n}\ \min_{x\in C} L(\alpha,x) \le 0. \tag{4}$$

The existence of an $x \in C$ satisfying all n inequalities $f_i(x) \le 0$, $1 \le i \le n$, is equivalent to the truth of the relation

$$\min_{x\in C}\ \max_{1\le i\le n} f_i(x) \le 0$$

or, which is the same,

$$\min_{x\in C}\ \max_{\alpha\in S_n} \sum_{i=1}^{n} \alpha_i f_i(x) = \min_{x\in C}\ \max_{\alpha\in S_n} L(\alpha,x) \le 0.$$

This relation can be obtained by Lemma 7 and relation (4) as follows

$$\min_{x\in C}\ \max_{\alpha\in S_n} L(\alpha,x) = \max_{\alpha\in S_n}\ \min_{x\in C} L(\alpha,x) \le 0. \qquad \square$$

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