REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 32 (2003) no. 1, pp. 11–20
ictp.acad.ro/jnaat

THE COMBINED SHEPARD–ABEL–GONCHAROV UNIVARIATE OPERATOR*

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Abstract. We extend the Shepard operator by combining it with the Abel-Goncharov univariate operator in order to increase the degree of exactness and to use some specific functionals. We study this combined operator and give some of its properties. We introduce the corresponding interpolation formula and study its remainder term.

MSC 2000. 65D05.

Keywords. Shepard-Abel-Goncharov operators, interpolation formula, degree of exactness, remainder term.

1. INTRODUCTION

1.1. The Shepard univariate operator. Recall first some results regarding the multivariate Shepard operator for the univariate case. Let f be a real valued function defined on $X \subset \mathbb{R}$ and $x_i \in X,\ i = 0, \ldots, N$, be some distinct points. The univariate Shepard operator is defined by

$$(Sf)(x) = \sum_{i=0}^{N} A_i(x) f(x_i), \tag{1}$$

where

$$A_i(x) = \frac{\prod\limits_{j=0,\, j\neq i}^{N} |x - x_j|^{\mu}}{\sum\limits_{k=0}^{N} \prod\limits_{j=0,\, j\neq k}^{N} |x - x_j|^{\mu}}, \tag{2}$$

with $\mu \in \mathbb{R}_+$ (see, e.g., [12]). The basis functions A_i may be written in barycentric form

$$A_i(x) = \frac{|x - x_i|^{-\mu}}{\sum\limits_{k=0}^{n} |x - x_k|^{-\mu}}.$$

It is easy to check that

$$A_i(x_v) = \delta_{iv}, \quad i, v = 0, \ldots, N,$$

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*This work was supported by CNCSIS Grant no. 8/139, 2003.

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and

$$\sum_{i=0}^{N} A_i(x) = 1. \tag{3}$$

The main properties of the operator S are:

- The interpolation property
$$(Sf)(x_i) = f(x_i), \quad i = 0, \ldots, N;$$
- The degree of exactness is $\operatorname{gex}(S) = 0$.

The goal when extending the operator S by combining with other operators is to increase the degree of exactness and to use other sets of functionals. Let $\Lambda := \{\lambda_i \mid i = 0, \ldots, N\}$ be a set of functionals and let P be the corresponding interpolation operator. We consider that $\Lambda_i \subset \Lambda$ are the subsets associated to the functionals λ_i, $i = 0, \ldots, N$. We have $\bigcup_{i=0}^{N} \Lambda_i = \Lambda$ and $\Lambda_i \bigcap \Lambda_j \neq \emptyset$, excepting the case $\Lambda_i = \{\lambda_i\}$, $i = 0, \ldots, N$, when $\Lambda_i \bigcap \Lambda_j = \emptyset$, for $i \neq j$. We associate the interpolation operator P_i to each subset Λ_i, $i = 0, \ldots, N$.

The operator S_P defined by

$$(S_P f)(x) = \sum_{i=0}^{N} A_i(x)(P_i f)(x) \tag{4}$$

is the combined operator of S and P (see, e.g., [12]).

Remark 1. As noted in [12], if P_i, $i = 0, \ldots, N$, are linear operators, then S_P is a linear operator. □

Remark 2. [12]. Let P_i, $i = 0, \ldots, N$, be some arbitrary linear operators. If $\operatorname{gex}(P_i) = r_i$, $i = 0, \ldots, N$, then

$$\operatorname{gex}(S_P) = r_m := \min\{r_0, \ldots, r_N\}. \qquad \square$$

Assume that Λ is a set of Birkhoff type functional, i.e.,

$$\Lambda_B = \{\lambda_{kj} \mid \lambda_{kj} f = f^{(j)}(x_k),\ j \in I_k,\ k = 1, \ldots, N\},$$

where $I_k \subseteq \{0, 1, \ldots, r_k\}$, for $r_k \in \mathbb{N}$. Denote $r_M = \max\{r_1, \ldots, r_N\}$.

Remark 3. [12]. If $\mu > r_M$ then $\lambda_{kj}(S_P f) = \lambda_{kj}(f)$, $j \in I_k$, $k = 0, \ldots, N$, where P is the interpolation operator corresponding to the set Λ_B.

In the proof of this result the following relations are used:

$$\begin{aligned} &A_i^{(v)}(x_k) = 0, \quad v \in I_k,\ k = 0, \ldots, N,\ k \neq i, \\ &A_i^{(v)}(x_i) = 0, \quad v \in I_i,\ v \geq 1, \\ &A_i^{(j)}(x_i) = 1. \end{aligned} \tag{5}$$

□

1.2. **The Abel–Goncharov univariate operator.** Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$, and $f : [a,b] \to \mathbb{R}$ be a function having the first n derivatives $f^{(i)}$, $i = 1, 2, \ldots, n$. Given the nodes $x_i \in [a,b]$, $0 \le i \le n$, and the values $f^{(i)}(x_i)$, $0 \le i \le n$, we consider the Abel–Goncharov interpolation problem of finding a polynomial $P_n f$ of degree n such that (see, e.g., [8] and [10])

$$(P_n f)^{(i)}(x_i) = f^{(i)}(x_i), \quad 0 \le i \le n. \tag{6}$$

The determinant of this linear system

$$D = \begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 0 & 1! & 2x_1 & \ldots & nx_1^{n-1} \\ 0 & 0 & 2! & \ldots & n(n-1)x_2^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & n! \end{vmatrix} = 1 \cdot 1! \cdot 2! \cdot \ldots \cdot n!, \tag{7}$$

is always nonzero and the problem (6) has therefore a unique solution. The Abel–Goncharov interpolation polynomial $P_n f$ can be written in the form

$$(P_n f)(x) = \sum_{k=0}^{n} g_k(x) f^{(k)}(x_k),$$

where g_k, $k = 0, \ldots, n$ are called Goncharov polynomials of degree k [9], determined by the conditions

$$\begin{cases} g_k^{(s)}(x_s) = 0, & \text{if } k \neq s, \\ g_k^{(k)}(x) = 1. \end{cases}$$

According to [8], [9] and [10], we have:

$$\begin{aligned} g_0(x) &= 1, \\ g_1(x) &= x - x_0, \\ g_k(x) &= \int_{x_0}^{x} \mathrm{d}t_1 \int_{x_1}^{t_1} \mathrm{d}t_2 \cdots \int_{x_{k-1}}^{t_{k-1}} \mathrm{d}t_k \\ &= \tfrac{1}{k!}\Big[x^k - \sum_{j=0}^{k-1} g_j(x)\tbinom{k}{j} x_j^{k-j}\Big], \quad k = 2, \ldots, n. \end{aligned} \tag{8}$$

Remark 4. [10]. When all the nodes coincide, then the problem (6) is a Taylor interpolation problem and $P_n f$ takes the form

$$(P_n f)(x) = \sum_{k=0}^{n} \tfrac{(x-x_0)^k}{k!} f^{(k)}(x_0). \qquad \square$$

Regarding the degree of exactness, we obtain the following result.

Theorem 1. *The Abel–Goncharov operator P_n has the degree of exactness n, i.e.,*

$$\operatorname{dex}(P_n) = n.$$

Proof. It is easily seen that for the test functions $e_i(x) = x^i$, $x \in [a, b]$, we have

$$(P_n e_i)(x) = e_i(x),\ i = 0, \ldots, n,$$

while

$$(P_n e_{n+1})(x) = g_0(x)x_0^{n+1} + g_1(x)(n+1)x_1^n + \ldots + g_n(x)(n+1) \cdot \ldots \cdot 2x_n \neq e_{n+1}(x).$$

□

We obtain the following result regarding the remainder $R_n f$ of the Abel–Goncharov interpolation formula

$$f = P_n f + R_n f.$$

THEOREM 2. *If* $f \in H^{n+1}[a, b]$ *then*

$$(R_n f)(x) = \int_a^b \varphi_n(x, s) f^{(n+1)}(s) \mathrm{d}s,$$

with

$$\varphi_n(x, s) = \tfrac{(x-s)_+^n}{n!} - \sum_{k=0}^{n} g_k(x) \tfrac{(x_k - s)_+^{n-k}}{(n-k)!}. \tag{9}$$

Proof. From Theorem 1 we have that $\mathrm{gex}(P_n) = n$. By Peano's theorem we obtain

$$(R_n f)(x) = \int_a^b \varphi_n(x, s) f^{(n+1)}(s) \mathrm{d}s,$$

with

$$\varphi_n(\cdot, s) = R_n \left[\tfrac{(\cdot - s)_+^n}{n!} \right] = \tfrac{(\cdot - s)_+^n}{n!} - P_n \left[\tfrac{(\cdot - s)_+^n}{n!} \right].$$

For all $x \in [a, b]$ we have

$$\varphi_n(x, s) = \tfrac{(x-s)_+^n}{n!} - \sum_{k=0}^{n} g_k(x) \left[\tfrac{(x_k - s)_+^n}{n!} \right]^{(k)},$$

which, after some immediate manipulations, implies (9). □

2. THE COMBINED SHEPARD–ABEL–GONCHAROV UNIVARIATE OPERATOR

In this section we shall assume that there exists $f^{(i)}(x_i)$, $i = 0, \ldots, N$, on the set of $N+1$ pairwise distinct points $x_i \in [a, b]$, $0 \leq i \leq N$. Let us consider the set of linear functionals of Abel–Goncharov type:

$$\Lambda_{AG}(f) := \{\lambda_i(f) : \lambda_i(f) = f^{(i)}(x_i),\ i = 0, \ldots, N\}.$$

We attach to each node x_i, $i = 0, \ldots, N$, a set of nodes $X_{i,n}$, $n \in \mathbb{N}$, $n \leq N$, $i = 0, \ldots, N$, defined by

$$X_{i,n} = \{x_i, x_{i+1}, \ldots, x_{i+n}\} = \{x_{i+\upsilon} : \upsilon = 0, \ldots, n\}, \quad i = 0, \ldots, N, \tag{10}$$

where $x_{N+k+1} = x_k$, $k = 0, \ldots, n$.

We associate to each set of nodes $X_{i,n}$, $i = 0, \ldots, N$, the Abel–Goncharov interpolation operator, denoted P_i^n, $i = 0, \ldots, N$, corresponding to the set of functionals Λ_{AG}. The operators P_i^n, $i=0,\dots,N$, exist and are unique because the points of the sets $X_{i,n}$, $i=0,\dots,N$, are pairwise distinct so the determinant of the interpolation system of the form (7) is always different from zero. We have

$$(P_i^n f)^{(k)}(x_k) = f^{(k)}(x_k), \quad i \le k \le i+n,\, 0 \le i \le N. \tag{11}$$

Remark 5. The set of linear functional of Abel–Goncharov type, Λ_{AG}, is included in the set of linear functional of Birkhoff type. We notice that in case of the Abel–Goncharov interpolation we have the advantage that the determinant of the interpolation system of the form (7) is always different from zero, thus the interpolation polynomial always exists and is unique. □

We consider the Abel–Goncharov polynomials of degree n, associated to the sets of nodes $X_{i,n}$, $i=0,\dots,N$, and the sets of linear functionals of Abel–Goncharov type given by

$$(P_i^n f)(x) = \sum_{j=i}^{i+n} g_{j-i}^{\langle i\rangle}(x) f^{(j-i)}(x_j), \quad i=0,\dots,N, \tag{12}$$

with the Goncharov polynomials given by

$$\begin{aligned} g_0^{\langle i\rangle}(x) &= 1,\\ g_1^{\langle i\rangle}(x) &= x - x_i,\\ g_k^{\langle i\rangle}(x) &= \tfrac{1}{k!}\Big[x^k - \sum_{j=0}^{k-1} g_j(x)\tbinom{k}{j} x_{j+i}^{k-j}\Big], \quad k \ge 1. \end{aligned}$$

Theorem 3. *The Abel–Goncharov operators P_i^n, $i=0,\dots,N$, have the degree of exactness n, i.e.,*

$$\operatorname{dex}(P_i^n) = n, \quad i=0,\dots,N. \tag{13}$$

The proof is obtained similarly to that of Theorem 1.

Remark 6. If we consider the sets X_{i,n_i}, $i=0,\dots,N$, of the form (10) such that each of them has n_i+1 elements, $n_i \in \mathbb{N}$, then

$$\operatorname{dex}(P_i^n) = n_i, \quad i=0,\dots,N.$$ □

We denote by S_n^{AG} the Shepard operator of Abel–Goncharov type, given by

$$(S_n^{AG} f)(x) = \sum_{i=0}^{N} A_i(x)(P_i^n f)(x),$$

where A_i, $i=0,\dots,N$, are given by (2) and P_i^n, $i=0,\dots,N$, are given by (12). We call S_n^{AG} the combined Shepard–Abel–Goncharov operator.

Particular case. When all the nodes coincide, $x_0 = \ldots = x_N$, we obtain the Shepard operator of Taylor type:

$$(T_n^i f)(x) = (P_i^n f)(x) = \sum_{j=0}^{n} \tfrac{(x-x_i)^j}{j!} f^{(j)}(x_i), \quad i = 0, \ldots, N,$$

and the combined Shepard–Taylor operator given by

$$(ST_n f)(x) = \sum_{i=0}^{N} A_i(x)(T_n^i f)(x).$$

The main properties of the Shepard–Taylor operator ST_n are:

- For $\mu > n$

$$(ST_n f)^{(j)}(x_i) = f^{(j)}(x_i), \quad j = 0, \ldots, n, \; i = 0, \ldots, N;$$

- $\operatorname{gex}(ST_n) = n.$ □

Theorem 4. *The operator S_n^{AG} is linear.*

Proof. For arbitrary h_1, $h_2 : [a, b] \to \mathbb{R}$ and α, $\beta \in \mathbb{R}$, one gets

$$\begin{aligned} & S_n^{AG}(\alpha h_1 + \beta h_2)(x) = \\ & = \sum_{i=0}^{N} A_i(x) \sum_{j=i}^{i+n} g_{j-i}^{\langle i\rangle}(x)(\alpha h_1 + \beta h_2)^{(j-i)}(x_j) \\ & = \alpha \sum_{i=0}^{N} A_i(x) \sum_{j=i}^{i+n} g_{j-i}^{\langle i\rangle}(x) h_1^{(j-i)}(x_j) + \beta \sum_{i=0}^{N} A_i(x) \sum_{j=i}^{i+n} g_{j-i}^{\langle i\rangle}(x) h_2{}^{(j-i)}(x_j) \\ & = \alpha S_n^{AG}(h_1)(x) + \beta S_n^{AG}(h_2)(x), \end{aligned}$$

which shows the linearity of S_n^{AG}. □

Theorem 5. *If $\mu > N$, the operator S_n^{AG} has the interpolation property:*

$$(S_n^{AG} f)^{(k)}(x_k) = f^{(k)}(x_k), \quad 0 \le k \le N. \tag{14}$$

Proof. Taking into account that $\mu > N$, we have

$$(S_n^{AG} f)^{(k)}(x_k) = \sum_{i=0}^{N} \sum_{\nu=0}^{k} \tbinom{k}{\nu} A_i^{(\nu)}(x_k)(P_i^n f)^{(k-\nu)}(x_k).$$

By (5) and (11) we obtain (14). □

Theorem 6. *$S_n^{AG} f = f$, for all $f \in \mathbb{P}_n$, where $\mathbb{P}_n$ is the set of polynomials of degree at most n.*

Proof. From Theorem 2 we have

$$\operatorname{gex}(S_n^{AG}) = \min \{ \operatorname{gex}(P_i^n) \mid i = 0, \ldots, N \},$$

and, taking into account (13), we obtain $\operatorname{gex}(S_n^{AG}) = n$. □

Remark 7. If the conditions of Remark 6 are verified, i.e., $\operatorname{gex}(P_i^n)=n_i$, $i=0,\dots,N$, then by Theorem 2 we have

$$\operatorname{gex}(S_n^{AG})=\min_{i\in\{0,\dots,N\}} n_i. \qquad \square$$

The Shepard–Abel–Goncharov interpolation formula is

$$f=S_n^{AG}f+R_n^{AG}f,$$

where $R_n^{AG}f$ denotes the remainder.

When all the nodes coincide we have the Shepard–Taylor interpolation formula and the following result is known.

Theorem 7. [4]. *If $f\in H^{n+1}[a,b]$ and $x_0=x_1=\ldots=x_N$, then*

$$(R_n^{AG}f)(x)=\int_a^b \varphi_n(x,s)f^{(n+1)}(s)\mathrm{d}s,$$

where

$$\varphi_n(x,s)=\tfrac{1}{n!}\Big\{(x-s)_+^n-\sum_{i=0}^N A_i(x)\big[(x_i-s)_+^0(x-x_i)+(x_i-x)_+\big]^n\Big\}.$$

Theorem 8. *If $f\in H^{n+1}[a,b]$ then*

$$(R_n^{AG}f)(x)=\int_a^b \varphi_n(x,s)f^{(n+1)}(s)\mathrm{d}s,$$

with

$$\varphi_n(x,s)=\tfrac{(x-s)_+^n}{n!}-\sum_{i=0}^N A_i(x)\sum_{j=i}^{i+n} g_{j-i}^{\langle i\rangle}(x)\tfrac{(x_j-s)_+^{n-j+i}}{(n-j+i)!}. \tag{15}$$

Proof. Theorem 6 implies $\operatorname{gex}(S_n^{AG})=n$. Applying the Peano's theorem, we obtain

$$(R_n^{AG}f)(x)=\int_a^b \varphi_n(x,s)f^{(n+1)}(s)\mathrm{d}s,$$

with

$$\varphi_n(\cdot,s)=R_n^{AG}\left[\tfrac{(\cdot-s)_+^n}{n!}\right]=\tfrac{(\cdot-s)_+^n}{n!}-\sum_{i=0}^N A_i(\cdot)P_i^n\left[\tfrac{(\cdot-s)_+^n}{n!}\right].$$

For all $x\in[a,b]$ we have

$$\varphi_n(x,s)=\tfrac{(x-s)_+^n}{n!}-\sum_{i=0}^N A_i(x)\sum_{j=i}^{i+n} g_{j-i}^{\langle i\rangle}(x)\left[\tfrac{(x_j-s)_+^n}{n!}\right]^{(j-i)},$$

and finally (15). $\square$

Particular case. We consider $n = 1$. We have the corresponding Shepard–Abel–Goncharov operator given by

$$\begin{aligned}(S_1^{AG}f)(x) &= \sum_{i=0}^{N} A_i(x)(P_i^1 f)\\ &= \sum_{i=0}^{N} A_i(x)\Big[g_0^{\langle i\rangle}(x)f(x_i) + g_1^{\langle i\rangle}(x)f'(x_{i+1})\Big].\end{aligned}$$

The interpolation formula is

$$f = S_1^{AG}f + R_1^{AG}f,$$

where $R_1^{AG}f$ is the remainder, which according with Theorem 8 has the following form:

$$(R_1^{AG}f)(x) = \int_a^b \varphi_1(x,s)f''(s)\mathrm{d}s,$$

with

$$\varphi_1(x,s) = (x-s)_+ - \sum_{i=0}^{N} A_i(x)\Big[g_0^{\langle i\rangle}(x)(x_i-s)_+ + g_1^{\langle i\rangle}(x)(x_{i+1}-s)_+^0\Big]. \quad \square$$

Example 1. Consider $f : [0,4] \to \mathbb{R}$, $f(x) = 3\sin\frac{\pi x}{4}$, and the nodes $x_i = i$, $i = 0, \ldots, 4$. Figure 1 shows the Shepard approximation corresponding to these data, while Figure 2 shows the Shepard–Abel–Goncharov approximation. The figures were drawn using Matlab. $\square$

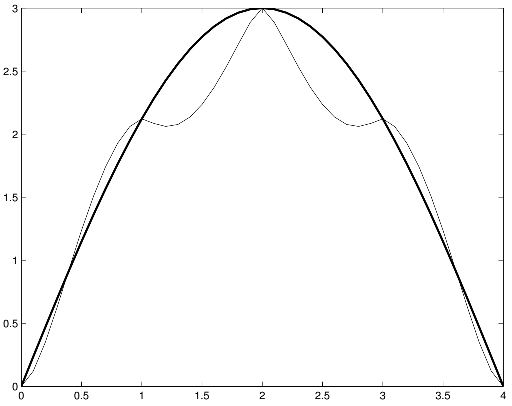

Fig. 1. Shepard interpolation.

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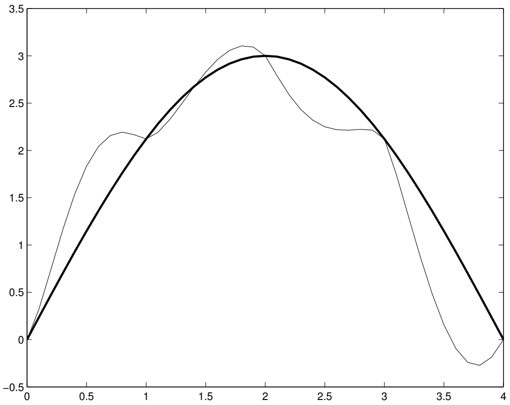

Fig. 2. Shepard-Abel-Goncharov interpolation.

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Received by the editors: November 9, 2002.