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THE APPROXIMATION OF THE SOLUTIONS OF EQUATIONS USING APPROXIMANT SEQUENCES

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Abstract. We intend to characterize the convergence of a certain sequence that belongs to a subset of a Banach space towards the solution of an equation obtained by the annulment of a nonlinear mapping that is defined on this subset and that takes values in another linear normed space. This mapping has a Fréchet derivative of a certain order which verifies the Lipschitz condition. We can establish some conditions that are enough both for the existence of the equation's solution and for a speed of convergence of a certain order for the approximant sequence.

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1. INTRODUCTION

One of the most often used methods for the approximation of the solution of an equation is that of constructing a sequence that is convergent to that solution.

Let us consider X and Y two normed linear spaces, their norms $\|\cdot\|_X$ and respectively $\|\cdot\|_Y$, a set $D \subseteq X$, a function $f : D \to Y$, θ_Y the null element of the space Y and, using these elements, the equation:

$$f(x) = \theta_Y. \tag{1}$$

To clarify these notions, we consider:

DEFINITION 1. *In addition to the data above, let us also consider $p \in \mathbb{N}$, not null and $(x_n)_{n\in\mathbb{N}} \subseteq D$. We say that the sequence is an approximant sequence of the order p of a solution of the equation* (1), *if there exist $\alpha, \beta \geq 0$ so that for any $n \in \mathbb{N}$ we have:*

$$\begin{aligned} &\|f(x_{n+1})\|_Y \leq \alpha \|f(x_n)\|_Y^p, \\ &\|x_{n+1} - x_n\|_X \leq \beta \|f(x_n)\|_Y. \end{aligned} \tag{2}$$

As we showed in [3] and [4], if $(x_n)_{n\in\mathbb{N}}$ is an approximant sequence of the order p, $p \geq 2$, X is a Banach space, $f : D \to Y$ is continuous, and the

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constants α and β from Definition 1 are chosen so that:

$$\rho_0 = \alpha^{\frac{1}{p-1}} \left\| f(x_0) \right\|_Y , \qquad S(x_0, \delta) = \left\{ x \in X : \ \|x - x_0\|_X \leq \delta \right\} \subseteq D \tag{3}$$

with:

$$\delta = \frac{\beta \alpha^{\frac{1}{p-1}}}{1 - \rho_0^{p-1}},$$

then the approximant sequence is convergent towards the element $\bar{x}$ which, together with all the terms of the sequence $(x_n)_{n \in \mathbb{N}}$ is placed in the ball $S(x_0, \delta)$ and $\bar{x}$ is a solution of the equation (1). For any $n \in \mathbb{N}$ we have the following inequalities:

$$\|x_{n-1} - x_n\|_X \leq \beta \alpha^{\frac{1}{p-1}} \rho_0^{p^n}, \qquad \|\bar{x} - x_n\| \leq \frac{\beta \alpha^{\frac{1}{p-1}} \rho_0^{p^n}}{1 - \rho_0^{p^n \, (p-1)}} \tag{4}$$

These inequalities justify the fact of calling it an approximant sequence of the order p.

In order to verify the inequalities (4) as well as the affirmations preceding them we have to make the inequalities (2) true. But this often proves to be difficult, and this is the reason for which we will try to replace them with more practical conditions. Nevertheless we will consider that the function $f : D \to Y$ admits Fréchet derivatives up to the order p included.

As a series of iterative methods known in practice use the inverse of the Fréchet derivative of the first order of the mapping $f'(x_n)^{-1}$, an unpractical condition, as the existence of this mapping implies solving the linear equation $f'(x_n)h = q$; $h \in X$, $q \in Y$, we will try to eliminate the conditions about the inverse of the Fréchet derivative from the hypothesis, but we will try to demonstrate this existence.

From the results that have inspired this research we will mention primarily the well-known theorem of L. V. Kantorovich for the case when the approximant sequence $(x_n)_{n \in \mathbb{N}}$ is generated by the Newton–Kantorovich method [5], [6]. In this case the existence of the mapping $f'(x)^{-1} \in (Y, X)^*$ is supposed only for $x = x_0$, as this is the initial point of the iterative method. In what the convergence of the same method is concerned, we also mention the result obtained by Mysovski, I. P. [7], where from a certain point of view the conditions of the convergence are simpler, but the existence of the mapping $f'(x)^{-1}$ and of a constant $M > 0$ satisfying the inequality $\|f'(x)^{-1}\| \leq M$ for any x an element of a certain ball centered in the initial element x_0 is imposed. Then Păvăloiu, I., in [8] and [9], generalizes these results for the convergence of a sequence generated by the relation of recurrence:

$$x_{n+1} = Q(x_n), \tag{5}$$

where $Q : X \to X$ verifies certain conditions and $n \in \mathbb{N}$. In the result obtained by Păvăloiu, I., Mysovski's condition mentioned above does not appear explicitly, but the use of the result in concrete cases makes it necessary.

We will proceed in the same way as in our papers [1], [2].

2. MAIN RESULT

Let us now note by $(X^p, Y)^*$ the set of p-linear and continuous mappings defined on $X^p = X \times \cdots \times X$ (the p times Cartesian product), taking values in Y.

The fact that the mapping $f^{(p)} : D \to (X^p, Y)^*$ verifies the Lipschitz condition is resumed to the existence of the constant $L > 0$, so that for any $x, y \in D$ we have:

$$\left\| f^{(p)}(x) - f^{(p)}(y) \right\| \le L \left\| x - y \right\|_X , \tag{6}$$

so that L will be called Lipschitz constant.

We can easily deduce the following inequality for any $x, y \in D$ we have:

$$\left\| f(x) - f(y) - \sum_{i=1}^{p} \tfrac{1}{i!} f^{(i)}(y)(x-y)^i \right\|_Y \le \tfrac{L}{(p+1)!} \left\| x - y \right\|_X^{p+1} . \tag{7}$$

Then if we take $x_0 \in D$ and $\delta > 0$ so that:

$$S(x_0, \delta) = \left\{ x \in X : \left\| x - x_0 \right\|_X \le \delta \right\} \subseteq D$$

and we define the numbers $L_0, \ldots, L_p, L_{p+1} > 0$ through $L_{p+1} = L$ and for any $k \in \{0, 1, \ldots, p\}$ we have:

$$L_k = \left\| f^{(k)}(x_0) \right\| + L_{k+1}\delta, \tag{8}$$

then for any $x \in S(x_0, \delta)$ and $k \in \{0, 1, \ldots, p\}$ we have:

$$\left\| f^{(k)}(x) \right\| \le L_{k+1}\delta \tag{9}$$

and for any $x, y \in S(x_0, \delta)$ and $k = 1, 2, \ldots, p+1$ we have:

$$\left\| f^{(k-1)}(x) - f^{(k-1)}(y) \right\| \le L_k \left\| x - y \right\|_X .$$

Under the conditions mentioned above, the following takes place:

THEOREM 2. *In addition to the data above we consider* $p \in \mathbb{N}$, $\delta > 0$, $(x_n)_{n \in \mathbb{N}} \subseteq D$.

Assume that:

i) X *is a Banach space and* $S(x_0, \delta) \subseteq D$, $S(x_0, \delta)$ *representing the ball with the center* x_0 *and radius* δ;

ii) *the function* $f : D \to Y$ *admits Fréchet derivatives up to the order* p *including it, and for* $f^{(p)} : D \to (X^p, Y)^*$, $L > 0$ *such that the following inequality holds for any* $x, y \in D$ *we have the inequality:*

$$\left\| f^{(p)}(x) - f^{(p)}(y) \right\| \le L \left\| x - y \right\|_Y ;$$

iii) *there exist the numbers* $a, b \geq 0$ *so that for any* $n \in \mathbb{N}$ *we have:*

$$\left\| f(x_n) + \sum_{i=1}^{p} \tfrac{1}{i!} f^{(i)}(x_n)(x_{n+1} - x_n)^i \right\|_Y \leq a \left\| f(x_n) \right\|_Y^{p+1} \tag{10}$$

and

$$\left\| f'(x_n)(x_{n+1} - x_n) \right\|_Y \leq b \left\| f(x_n) \right\|_Y ; \tag{11}$$

iv) *the mapping* $f'(x_0) \in (X, Y)^*$ *is invertible;*

v) *using the notation:*

$$\begin{gathered} \rho_0 = \left\| f(x_0) \right\|_Y, \quad B_0 = \left\| f'(x_0)^{-1} \right\|, \quad h_0 = bL_2 B_0^2 \rho_0, \\ M = B_0 \mathrm{e}^{1+2^{-2p+3}}, \quad \alpha = a + L \tfrac{(bM)^{p+1}}{(p+1)!} \end{gathered} \tag{12}$$

suppose the following inequalities hold:

$$h_0 \leq \tfrac{1}{2}, \quad \alpha^{\frac{1}{p}} \rho_0 < \tfrac{1}{4}, \quad \delta \geq \tfrac{bM\rho_0}{1 - \alpha\rho_0^p}, \tag{13}$$

then:

j) $x_n \in S(x_0, \delta)$, $f'(x_0)^{-1}$ *exists and* $\left\| f'(x_0)^{-1} \right\| \geq M$, *for any* $n \in \mathbb{N}$;

jj) *equation* (1) *admits a solution* $\bar{x} \in S(x_0, \delta)$;

jjj) *the sequence* $(x_n)_{n\in\mathbb{N}}$ *is an approximant sequence of the order* $p+1$ *of this solution of the equation* (1);

jv) *the following estimates hold for any* $n \in \mathbb{N}$:

$$\max \left\{ \left\| f(x_n) \right\|_Y, \tfrac{1}{Mb} \left\| x_{n+1} - x_n \right\|_X \right\} \leq \alpha^{\frac{(p+1)^n - 1}{p}} \left\| f(x_0) \right\|_Y^{(p+1)^n} \tag{14}$$

and

$$\left\| \bar{x} - x_n \right\|_X \leq Mb \frac{\alpha^{\frac{(p+1)^n - 1}{p}} \left\| f(x_0) \right\|_Y^{(p+1)^n}}{1 - \left(\alpha \left\| f(x_0) \right\|_Y^p \right)^{(p+1)^n}}. \tag{15}$$

Proof. From the invertibility of the mapping $f'(x_0) \in (X, Y)^*$ we clearly deduce that $\left\| f'(x_0) \right\|, \left\| f'(x_0)^{-1} \right\| > 0$.

Let the sequences $(\rho_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ be so that $\rho_0 = \left\| f'(x_0) \right\|$, $B_0 = \left\| f'(x_0)^{-1} \right\|$, and for any $n \in \mathbb{N}$, we have:

$$h_n = bL_2 B_0^2 \rho_n, \quad \rho_{n+1} = \alpha \rho_n^{p+1}, \quad B_{n+1} = \frac{B_n}{1 - h_n}.$$

We will show that for any $n \in \mathbb{N}$ the following statements are true:

a) $x_n \in S(x_0, \delta)$;

b) $f'(x_n)^{-1} \in (Y, X)^*$ exists, and $\left\| f'(x_n)^{-1} \right\| \leq B_n$;

c) $\left\| f(x_n) \right\|_Y \leq \rho_n = \alpha^{-\frac{1}{p}} \left(\alpha^{\frac{1}{p}} \rho_0 \right)^{(p+1)^n}$;

d) $h_n \leq \min \left\{ \frac{1}{2}, \beta^{-\frac{1}{p}} (\beta h_0)^{(p+1)^n} \right\}$, where $\beta = \frac{4}{(4h_0)^p}$;

e) $B_0 \leq B_n \leq M$.

Using the mathematical induction we notice that for $n = 0$ the statements a)–e) are evidently true from the hypotheses of the theorem with the notations we have introduced.

Let us suppose that for any $n \le k$ the assertions a)–e) are true, and let us demonstrate them for $n = k + 1$.

a) We notice that for any $n \in \mathbb{N}$, $n \le k$ we have:

$$\|x_{n+1} - x_n\|_X \le \|f'(x_n)^{-1} f'(x_n)(x_{n+1} - x_n)\|_Y \le Mb\alpha^{-\frac{1}{p}} (\alpha^{\frac{1}{p}} \rho_0)^{(p+1)^n},$$

from where:

$$\|x_{k+1} - x_0\|_X \le Mb\alpha^{-\frac{1}{p}} \rho_0 \sum_{n=0}^{k} (\alpha^{\frac{1}{p}} \rho_0)^{(p+1)^n - 1} \le \frac{Mb\rho_0}{1 - \alpha\rho_0^p} \le \delta$$

which shows that $x_{k+1} \in S(x_0, \delta)$.

b) Let

$$H_K = f'(x_k)^{-1} (f'(x_k) - f'(x_{k+1})) \in (X, X)^*,$$

its existence and its belonging to $(X, X)^*$ being guaranteed by the hypothesis of the induction. It is obvious that:

$$\|H_k\| \le B_k L_2 \|x_{k+1} - x_k\|_X \le bL_2 B_k^2 \rho_k = h_k \le \tfrac{1}{2} < 1,$$

and according to the Banach theorem we deduce that $(I_X - H_k)^{-1} \in (X, X)^*$ and:

$$\| (I_X - H_k)^{-1} \| \le \frac{1}{1 - \|H_k\|} \le \frac{1}{1 - h_k},$$

where $I_X : X \to X$ represents the identical mapping of the space X.

Obviously $f'(x_{k+1}) = f'(x_k)(I_X - H_k)$ and because $f'(x_k)^{-1} \in (Y, X)^*$ exists, the mapping $f'(x_{k+1})^{-1} = (I_X - H_k)^{-1} f'(x_k)^{-1}$exist as well and:

$$\|f'(x_{k+1})^{-1}\| \le \|(I_X - H_k)^{-1}\| \cdot \|f'(x_k)^{-1}\| \le \frac{B_k}{1 - h_k} = B_{k+1}.$$

c) Clearly:

$$\begin{aligned}
\|f(x_{k+1})\|_Y \le & \left\| f(x_{k+1}) - f(x_k) - \sum_{i=1}^{p} \tfrac{1}{i!} f^{(i)}(x_k)(x_{k+1} - x_k)^i \right\|_Y \\
& + \left\| f(x_k) + \sum_{i=1}^{p} \tfrac{1}{i!} f^{(i)}(x_k)(x_{k+1} - x_k)^i \right\|_Y \le \\
& \le \left[a + \tfrac{L(Mb)^{p+1}}{(p+1)!} \right] \|f(x_k)\|_Y^{p+1} \\
& \le \alpha \rho_k^{p+1} \\
& = \rho_{k+1},
\end{aligned}$$

as $\alpha^{\frac{1}{p}} \rho_{k+1} = (\alpha^{\frac{1}{p}} \rho_k)^{p+1}$, so $\rho_{k+1} = \alpha^{-\frac{1}{p}} (\alpha^{\frac{1}{p}} \rho_0)^{(p+1)^{k+1}}$.

d) We have the equalities:

$$h_{k+1} = L_2 b B_{k+1}^2 \rho_{k+1} = L_2 b \alpha \rho_k^{p+1} \left(\tfrac{B_k}{1-h_k}\right)^2 = \alpha \rho_k^p \tfrac{h_k}{(1-h_k)^2}.$$

Since $h_k \le \frac{1}{2}$ and $\rho_k < \rho_0$, we have $h_{k+1} \le 2\alpha\rho_0^p$, so $h_{k+1} \le \frac{1}{2}$.
Also:

$$h_{k+1} = \frac{\alpha h_k}{(1-h_k)^2} \cdot \frac{h_k^p}{(bL_2 B_k^2)^p} = \frac{\alpha}{(L_2 b)^p} \cdot \frac{1}{B_k^{2p}} \cdot \frac{(h_k)^{p+1}}{(1-h_k)^2}.$$

From $B_k \ge B_0$ and $\frac{1}{(1-h_k)^2} \le 4$ we deduce that:

$$h_{k+1} \le \frac{4\alpha h_k^{p+1}}{(bL_2)^p\, B_0^{2p}} < \frac{4h_k^{p+1}}{4^p \rho_0^p \left(bL_2 B_0^2\right)^p} = \beta h_k^{p+1}$$

and then, it the same way as in the proof of c) we deduce that $h_{k+1} = \beta^{-\frac{1}{p}} \left(\beta^{\frac{1}{p}} h_0\right)^{(p+1)^{k+1}}$.

e) Because of the relation:

$$B_{k+1} = \frac{B_k}{1-h_k}$$

and from the condition $h_k \in]0, \frac{1}{2}]$ which implies $\frac{1}{1-h_k} \ge 1$, $B_{k+1} \ge B_k$ from where $B_{k+1} \ge B_0$.

As $\beta^{\frac{1}{p}} h_0 = \frac{4^{1/p}}{4} \le 1$ we deduce that:

$$\max_{n\in\mathbb{N}} \left\{\beta^{-\frac{1}{p}} \left(\beta^{\frac{1}{p}} h_0\right)^{(p+1)^n}\right\} = \beta^{-\frac{1}{p}} \left(\beta^{\frac{1}{p}} h_0\right) = h_0$$

and the same initial relation implies:

$$\begin{aligned} B_{k+1} =& \frac{B_0}{(1-h_0)(1-h_1)\dots(1-h_k)} \\ \le& B_0 \Big[1 + \tfrac{1}{k+1} \sum_{i=0}^{k} \tfrac{h_i}{1-h_i}\Big]^{k+1} \\ \le& B_0 \Big[1 + \tfrac{1}{(k+1)(1-h_0)} \sum_{i=0}^{k} h_i\Big]^{k+1}. \end{aligned}$$

For any $k \in \mathbb{N}$ we have:

$$h_{k+1} = \frac{\alpha h_k \rho_k^p}{(1-h_k)^2} \le 2\left(\alpha^{\frac{1}{p}} \rho_0\right)^{p(p+1)^k}$$

and:

$$\begin{aligned}\sum_{i=0}^{k} h_i =& h_0 + 2\sum_{i=1}^{k}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{p(p+1)^{i-1}}\\ <& h_0 + 2\alpha\rho_0^p\sum_{i=1}^{k}\left(\alpha\rho_0^p\right)^{i-1}\\ <& h_0 + \tfrac{2\alpha\rho_0^p}{1-\alpha\rho_0^{p^2}}\\ <& \tfrac{1}{2} + \tfrac{2^{2p^2-2p+1}}{2^{2p^2-1}}\\ =& \tfrac{1}{2} + 2^{-2p+2}.\end{aligned}$$

So, from $h_0 < \frac{1}{2}$, we have:

$$B_{k+1} \le B_0\left(1 + \tfrac{1+2^{-2p+3}}{k+1}\right)^{k+1} \le B_0 \exp\left(1 + 2^{-2p+3}\right) = M.$$

From the above we deduce that the statements a)–e) are true for $n = k+1$. According to the principle of mathematical induction these statements are true for any $n \in \mathbb{N}$.

Now we will deduce that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, because:

$$\begin{aligned}\|x_{n+m} - x_n\|_X <& Mb\alpha^{-\frac{1}{p}}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}\sum_{i=1}^{m}\left(\alpha\rho_0^p\right)^{(p+1)^{i-1}}\\ <& \frac{Mb\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}}{\alpha^{\frac{1}{p}}\left[1-(\alpha\rho_0)^{(p+1)^n}\right]}.\end{aligned}$$

The last inequality and the condition $\alpha^{\frac{1}{p}}\rho_0 < \frac{1}{4} < 1$, determine the fact that $(x_n)_{n\in\mathbb{N}}$ is a fundamental sequence in the Banach space X, so $(x_n)_{n\in\mathbb{N}}$ is convergent. If we note $\bar{x} = \lim_{n\to\infty} x_n \in X$ and if we make so that $m \to \infty$ in the previous inequality we deduce the inequality (15), from where for $n = 0$ we can deduce:

$$\|\bar{x} - x_0\|_X \le \frac{bM\rho_0}{1-\alpha\rho_0^p} \le \delta,$$

so $\bar{x} \in S(x_0, \delta)$.

From:

$$\|f(x_n)\|_Y \le \alpha^{-\frac{1}{p}}\left(\alpha^{\frac{1}{p}}\rho_0\right)^{(p+1)^n}$$

and the condition $\alpha^{\frac{1}{p}}\rho_0 < 1$ we deduce that $\lim_{n\to\infty}\|f(x_n)\|_Y = 0$, from where $f(\bar{x}) = \theta_Y$, so $\bar{x}$ is a solution of the equation (1).

The inequalities:

$$\begin{aligned}\|x_{n+1} - x_n\|_X &\le Mb\,\|f(x_n)\|_Y\,,\\ \|f(x_{n+1})\|_Y &\le \alpha\,\|f(x_n)\|_Y^{p+1}\,,\end{aligned}$$

show that the sequence $(x_n)_{n\in N}$ is an approximant sequence of the order p for the solution $\bar{x}$. In this way the theorem is proved. $\square$

3. SPECIAL CASE

Now we will see how Theorem 2 is applied in the case of particular process of approximation.

Let us first suppose that the function $f : D \to Y$ admits for any $x \in D$ a Fréchet derivative of the first order, an $L > 0$ exists so that:

$$\|f'(x) - f'(y)\| \leq L\|x - y\|_X$$

for any $x, y \in D$, and the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ verifies for any $n \in \mathbb{N}$ the equality:

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y. \tag{16}$$

Obviously, if for any $n \in \mathbb{N}$, $f'(x_n)^{-1}$ exists, the relation (16) is equivalent to:

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \tag{17}$$

form under which the Newton–Kantorovich method is well known. But the form (16) will be one of the conclusions of the statement that will be established.

It is clear that the inequalities (10) and (11) of the hypothesis iii) of Theorem 2 are verified for $a = 0$ and $b = 1$.

In this case $p = 1$, $L_2 = L$, $h_0 = 2LB_0^2\rho_0$, $\alpha = \frac{LM^2}{2}$, $M = \|f'(x_0)^{-1}\|\mathrm{e}^3$, and thus the inequality of hypothesis v) of Theorem 2 becomes $\rho_0 < \frac{1}{4}$.

As $\alpha\rho_0 = \frac{LM^2h_0}{4LB_0^2} = \frac{\mathrm{e}^9h_0}{4}$, we need the condition $h_0 < \frac{1}{\mathrm{e}^9}$ or $B_0^2\rho_0 < \frac{1}{2\mathrm{e}^9L}$, condition that evidently also implies $h_0 \leq \frac{1}{2}$.

In what the radius of the ball on which the properties take place is concerned, it verifies the inequality $\delta \geq \frac{M\rho_0}{1-\alpha\rho_0}$.

As $\alpha\rho_0 < \frac{1}{4}$ we deduce that $\frac{1}{1-\alpha\rho_0} < \frac{1}{4}$ and so if $\delta \geq \frac{3M\rho_0}{4}$ the requirement is fulfilled. Also, $M = \|f'(x_0)^{-1}\|\mathrm{e}^3$.

In this way we have the following:

Corollary 3. *We consider the same elements as in Theorem* 1. *If the hypotheses* i), ii) *and* iv) *of this theorem are verified for* $p = 1$, *in addition the sequence verifies, for any* $n \in \mathbb{N}$, *the equalities:*

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y,$$

and the initial point $x_0 \in D$ *verifies the inequality:*

$$\|f'(x_0)^{-1}\|^2 \|f(x_0)\|_Y < \frac{1}{2\mathrm{e}^9L},$$

then:

j) *$x_n \in S(x_0,\delta)$, the mapping $f'(x_n)^{-1} \in (Y,X)^*$ exists and:*

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \quad \|f'(x_n)^{-1}\| \le \|f'(x_0)^{-1}\| \mathrm{e}^3,$$

for any $n \in \mathbb{N}$;

jj) *equation* (1) *admits a solution $\bar{x} \in S(x_0,\delta)$;*

jjj) *the sequence $(x_n)_{n\in\mathbb{N}}$ is an approximant sequence of the second order of the solution $\bar{x}$ of this equation;*

jv) *the following evaluations take place:*

$$\max\left\{ \|f(x_n)\|_Y, \tfrac{1}{M}\|x_{n+1}-x_n\|_X \right\} \le \left(\tfrac{LM^2}{2}\right)^{2^n-1} \|f(x_n)\|_Y^{2^n},$$

$$\|\bar{x} - x_n\|_X \le \frac{M\rho_0\left(\rho_0\frac{LM^2}{2}\right)^{2^n}}{1-\left(\rho_0\frac{LM^2}{2}\right)^{2^n}},$$

where $M = \|f'(x_0)^{-1}\|e^3$ and $L > 0$ represent the Lipschitz constant of the mapping $f' : D \to (X,Y)^*$.

Another case in which Theorem 2 is applied is the case of the Chebyshev method, that is to be studied in a forthcoming paper.

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