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ON STABILITY CONDITIONS OF VECTOR l_∞-EXTREME COMBINATORIAL PROBLEM WITH PARETO PRINCIPLE OF OPTIMALITY*

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Abstract. We consider the multicriteria problem of combinatorial optimization with partial criteria of the kind MINMAX MODUL. The parameters of criteria are subject to "small" independent perturbations. The class of problems for which new Pareto optima do not appear, but some trajectories may lose optimality under those perturbations, is distinguished.

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1. INTRODUCTION

Various aspects of stability of scalar and vector problems of discrete optimization have been considered in many publications (see, for example, [1] and reviews [2]–[5]). In this article, we continue our investigations (see [6], [7]) of stability of vector (multicriteria) problems of discrete optimization with non-linear partial criteria. Necessary and sufficient conditions of stability of the vector combinatorial problem with partial criteria of the kind MINMAX MODUL are obtained for such type of stability that can be understood as a discrete analogue of upper semicontinuity (by Hausdorf) of the many-valued mapping that puts in correspondence the Pareto set with collection of parameters of the problem. Similar results were recently obtained in [8], [9] for the case of lower semicontinuity.

2. BASIC DEFINITIONS AND LEMMA

Let $E = \{e_1, e_2, \ldots, e_m\}$, $m \geq 2$, $T \subseteq 2^E \backslash \{\emptyset\}$, $|T| > 1$, A_i be the i-th row of matrix $A = [a_{ij}]_{n\times m} \in \mathbb{R}^{nm}$, $n \geq 1$, $N_n = \{1, 2, \ldots, n\}$, $N(t) = \{j \in N_m : e_j \in t\}$, $t \in T$. Vector criterion

$$f(t, A) = \big(f_1(t, A_1), f_2(t, A_2), \ldots, f_n(t, A_n)\big) \to \min_{t\in T}$$

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is defined on the set of trajectories T. The partial criteria are of the kind MINMAX MODUL:

$$f_i(t, A_i) = \max_{j \in N(t)} |a_{ij}| \to \min_{t \in T}, \quad i \in N_n.$$

Thereby the value of i-th partial criterion on trajectory t is Chebyshev norm l_∞ of the vector with length $|t|$, formed by the elements of row A_i corresponding to trajectory t. Therefore it is natural to call the problem of finding the Pareto set (the set of efficient trajectories) [10]

$$P^n(A) = \{t \in T : \ P^n(t, A) = \emptyset\},$$

where $P^n(t, A) = \{t' \in T : \ f(t, A) \geq f(t', A), \ f(t, A) \neq f(t', A)\}$, the vector l_∞-extreme trajectorial problem. As E and T are fixed, we denote the problem by $Z^n(A)$.

It is clear that $P^1(A)$ (where A is an m dimensional vector) is set of all optimal solutions of the scalar trajectorial problem $Z^1(A)$. The most of well-known problems on graph, boolean programming problems and many problems of the scheduling theory [2], [3] are instances of this scalar problem.

As usual [5]–[9], we will perturb the parameters of vector criterion $f(t, A)$ by the addition of matrix $A \in \mathbb{R}^{nm}$ with matrices from the set

$$\mathcal{B}(\varepsilon) = \{B \in \mathbb{R}^{nm} : \ \|B\| < \varepsilon\},$$

where $\varepsilon > 0$ is the limiting level of perturbations, $\| \cdot \|$ is norm l_∞ in space $\mathbb{R}^{nm}$, i.e.,

$$\|B\| = \max\{|b_{ij}| : (i, j) \in N_n \times N_m\}, \ B = [b_{ij}]_{n \times m}.$$

Problem $Z^n(A + B)$, obtained from the initial problem $Z^n(A)$ by addition of matrices A and $B \in \mathcal{B}(\varepsilon)$, is called perturbed, the matrix B is called perturbing.

Corresponding to [1], [5]–[7] and from the above said, the stability property of problem $Z^n(A)$ is when new efficient trajectories do not appear under "small" independent perturbations of the elements of matrix A. Therefore problem $Z^n(A)$ is stable if and only if

$$\exists \varepsilon > 0 \text{ s.t. } \ P^n(A) \supseteq P^n(A + B), \ \ \forall B \in \mathcal{B}(\varepsilon).$$

Evidently, problem $Z^n(A)$ is stable when equality $P^n(A) = T$ holds. Therefore we consider only those problems $Z^n(A)$ for which $\bar{P}^n(A) := T \backslash P^n(A)$ is not empty. Such a problem is called nontrivial.

It is evident that the nontrivial problem is stable if and only if

$$\forall t \in \bar{P}^n(A) \ \ \exists \varepsilon > 0 \text{ s.t. } \ t \in \bar{P}^n(A + B), \ \ \forall B \in \mathcal{B}(\varepsilon). \tag{1}$$

Let us assign

$$g(t, t', A) = f(t, A) - f(t', A), \qquad g_i(t, t', A_i) = f_i(t, A_i) - f_i(t', A_i),$$

$$f_i(\emptyset, A_i) = -\infty, \qquad 0_{(n)} = (0, 0, \ldots, 0) \in \mathbb{R}^n.$$

Lemma 1. *For any trajectories* $t,\ t' \in T,\ t \neq t'$ *and arbitrary index* $i \in N_n$, *the implication*

$$g_i(t, t'\backslash t, A_i) > 0 \ \Rightarrow\ \exists \varepsilon > 0 \text{ s.t. } g_i(t, t', A_i + B_i) \geq 0, \quad \forall B \in \mathcal{B}(\varepsilon),$$

is true.

Proof. If $t'\backslash t \neq \emptyset$, then by the continuity of functions $g_i(t, t'\backslash t, A_i)$ on set $\mathbb{R}^m$, taking into account $g_i(t, t'\backslash t, A_i) > 0$, we obtain

$$\exists \varepsilon > 0 \text{ s.t. } \ g_i\big(t, t'\backslash t, A_i + B_i\big) > 0, \quad \forall B \in \mathcal{B}(\varepsilon).$$

The following two cases are possible.

Case 1: $f_i(t', A_i + B_i) = f_i(t'\backslash t, A_i + B_i)$. Then we obtain

$$g_i(t, t', A_i + B_i) > 0.$$

Case 2: $f_i(t', A_i + B_i) = f_i(t' \cap t, A_i + B_i),\ t' \cap t \neq \emptyset$. The relations

$$g_i(t, t', A_i + B_i) = f_i(t, A_i + B_i) - f_i(t' \cap t, A_i + B_i) \geq 0$$

are evident in this case.

If $t'\backslash t = \emptyset$, then $t\backslash t' \neq \emptyset$, since $t' \neq t$. Therefore inequality

$$g_i(t, t', A_i + B_i) = \max\big\{f_i(t\backslash t', A_i + B_i),\ f_i(t', A_i + B_i)\big\} - f_i(t', A_i + B_i) \geq 0$$

holds for any matrix $B \in \mathbb{R}^{nm}$. □

3. THEOREM

Theorem 2. *Nontrivial problem* $Z^n(A)$, $n \geq 1$, *is stable if and only if the formula*

$$\text{(2)} \qquad \forall t \in \bar{P}^n(A) \ \ \exists t' \in P^n(A) \text{ s.t. } \ g_i(t, t'\backslash t, A_i) > 0, \quad \forall i \in N_n,$$

is true.

Proof. Sufficiency. Let $t \in \bar{P}^n(A)$ and there exists a trajectory $t' \in P^n(A)$ such that the inequality $g_i(i, t'\backslash t, A_i) > 0$ holds for any index $i \in N_n$. Then, by Lemma 1, we obtain

$$\forall i \in N_n,\ \exists \varepsilon_i > 0 \text{ s.t. } \quad g_i(t, t', A_i + B_i) \geq 0, \quad \forall B \in \mathcal{B}(\varepsilon_i).$$

Therefore the formula

$$\text{(3)} \qquad \forall i \in N_n, \quad g_i(t, t', A_i + B_i) \geq 0, \quad \forall B \in \mathcal{B}(\varepsilon),$$

is true, where $\varepsilon = \min\{\varepsilon_i :\ i \in N_n\}$.

Furthermore, on account of $t \in \bar{P}^n(A)$ and $t' \in P^n(A)$, there exists an index $k \in N_n$ such that $g_k(t, t', A_k) > 0$. Therefore, by the continuity of functions $g_k(t, t', A_k)$ on $\mathbb{R}^m$, we obtain

$$\text{(4)} \qquad \exists \varphi > 0 \text{ s.t. } \quad g_k(t, t', A_k + B_k) > 0, \quad \forall B \in \mathcal{B}(\varphi).$$

Combining (3) with (4) we conclude that

$$g(t, t', A + B) \geq 0_{(n)} \quad \text{and} \quad g(t, t', A + B) \neq 0_{(n)}, \quad \forall B \in \mathcal{B}(\psi),$$

where $\psi = \min\{\varepsilon, \varphi\}$.

Hence, for any perturbing matrix $B \in \mathcal{B}(\psi)$ we have $t' \in \pi(t, A+B)$, i.e., $t \in \bar{P}^n(A+B)$. Finally, on account of (1) we derive

$$P^n(A+B) \subseteq P^n(A), \quad \forall B \in \mathcal{B}(\psi),$$

i.e., problem $Z^n(A)$ is stable.

Necessity. Suppose the opposite. Let the problem $Z^n(A)$ be stable and

$$\exists t^0 \in \bar{P}^n(A) \ \ \forall t \in P^n(A) \ \ \exists k \in N_n \text{ s.t.} \quad g_k(t^0, t\backslash t^0, A_k) \leq 0. \tag{5}$$

Then $t\backslash t^0 \neq \emptyset$ for any trajectory $t \in P^n(A)$. Otherwise (on account of $f_k(t\backslash t^0, A_k) = -\infty$) the inequality $g_k(t^0, t\backslash t^0, A_k) \leq 0$ is false.

For any index $i \in N_n$ we put

$$T_i = \{t \in P^n(A) : \ g_i(t^0, t\backslash t^0, A_i) \leq 0\}.$$

Then we introduce the definition $I = \{i \in N_n : \ T_i \neq \emptyset\}$. It is evident that $I \neq \emptyset$ and, by virtue of (5), the equality

$$\bigcup_{i \in I} T_i = P^n(A) \tag{6}$$

is true.

Let $0 < \beta < \varepsilon$. Consider perturbing matrix $B^* = [b^*_{ij}]_{n\times m}$ with the elements defined by

$$b_{ij} = \begin{cases} \beta, & \text{if } i \in I, \ j \in N(E\backslash t^0), \ a_{ij} \geq 0; \\ -\beta, & \text{if } i \in I, \ j \in N(E\backslash t^0), \ a_{ij} < 0; \\ 0, & \text{otherwise } (i,j) \in N_n \times N_m. \end{cases}$$

Since $I \neq \emptyset$ and $N(E\backslash t^0) \neq \emptyset$, it is clear, that $B^* \in \mathcal{B}(\varepsilon)$ and $\|B^*\| = \beta$.

On account of (5), for a fixed trajectory $t \in P^n(A)$ there exists an index $k \in N_n$ such that

$$g_k(t^0, t\backslash t^0, A_k) \leq 0. \tag{7}$$

Note, that $k \in I$ by virtue of (6).

Let us show that the inequality

$$g_k(t, t^0, A_k + B^*_k) \geq \beta \tag{8}$$

is valid. On account of the structure of matrix B^*, since $k \in I$, we obtain

$$\begin{aligned} g_k(t, t^0, A_k + B^*_k) &= f_k(t, A_k + B^*_k) - f_k(t^0, A_k + B^*_k) \\ &= \max\{f_k(t\backslash t^0, A_k + B^*_k), f_k(t \cap t^0, A_k + B^*_k)\} - f_k(t^0, A_k) \\ &= \max\{f_k(t\backslash t^0, A_k) + \beta, f_k(t \cap t^0, A_k)\} - f_k(t^0, A_k). \end{aligned} \tag{9}$$

Further, by virtue of (7), we have

$$f_k(t\backslash t^0, A_k) \geq f_k(t^0, A_k) \geq f_k(t \cap t^0, A_k).$$

Thereby we derive

$$\max\{f_k(t\backslash t^0, A_k) + \beta, f_k(t \cap t^0, A_k)\} = f_k(t\backslash t^0, A_k) + \beta \geq f_k(t^0, A_k) + \beta.$$

Therefore, according to (9), we obtain that the inequality (8) is true.

Resuming the above, we conclude that the formula

$$\forall t \in P^n(A) \;\; \exists k \in N_n \text{ s.t. } g_k(t, t^0, A_i + B^*_k) > 0 \tag{10}$$

is valid.

If $t^0 \in P^n(A + B^*)$, then on account of $t^0 \in \bar{P}^n(A)$ the formula

$$\forall \varepsilon > 0 \;\; \exists B^* \in \mathcal{B}(\varepsilon) \text{ s.t. } \quad P^n(A + B^*) \nsubseteq P^n(A) \tag{11}$$

is evident. It follows that problem $Z^n(A)$ is not stable. We have obtained a contradiction.

Let $t^0 \in \bar{P}^n(A + B^*)$. Since $|P^n(A + B^*)| < \infty$, the set $P^n(A + B^*)$ is externally stable (see [11, p. 34]). Terefore there exists trajectory t^* such that

$$t^* \in P^n(A + B^*), \quad g(t^*, t^0, A + B^*) \leq 0_{(n)}, \quad g(t^*, t^0, A + B^*) \neq 0_{(n)}.$$

Thereby there is no index $i \in N_n$ such that $g_i(t^*, t^0, A_i + B^*_i) > 0$. Then $t^* \in \bar{P}^n(A)$ by virtue of (10). Hence, we see that formula (11) is valid. It implies the contradiction to the statement that problem $Z^n(A)$ is stable. □

4. COROLLARIES

Let us introduce the traditional Slater set, i.e. the set of weakly efficient trajectories [10]

$$S^n(A) = \{t \in T : \quad S^n(t, A) = \emptyset\},$$

where $S^n(t, A) = \{t' \in T : \; g_i(t, t', A_i) > 0, \; \forall i \in N_n\}$. Evidently, the formula $P^n(A) \subseteq S^n(A)$ is valid for any matrix $A \in \mathbb{R}^{nm}$.

Corollary 3. *A nontrivial problem $Z^n(A)$, $n \geq 1$, is stable if and only if one of the following two alternatives holds:*

$$P^n(A) = S^n(A), \tag{12}$$

$$\forall t \in S^n(A) \backslash P^n(A), \quad \exists t' \in P^n(A) \text{ s.t. } \quad g_i(t, t' \backslash t, A_i) > 0, \quad \forall i \in N_n. \tag{13}$$

Proof. Let us show at first that the formula

$$\forall t^0 \in \bar{S}^n(A), \quad \exists t' \in P^n(A) \;\text{ s.t. }\; g_i(t^0, t' \backslash t^0, A_i) > 0, \quad \forall i \in N_n, \tag{14}$$

where $\bar{S}^n(A) = T \backslash S^n(A)$, is true for any problem $Z^n(A)$.

Really, since $t^0 \in \bar{S}^n(A)$, taking into account the definition of set $S^n(A)$, we obtain

$$\exists t' \in T \;\text{ s.t. }\; g_i(t^0, t', A_i) > 0, \forall i \in N_n. \tag{15}$$

If $t' \in P^n(A)$ then, by virtue of (15),

$$f_i(t^0, A_i) > f_i(t', A_i) \geq f_i(t' \backslash t^0, A_i), \quad \forall i \in N_n.$$

We see that formula (14) is valid.

If $t' \in \bar{P}^n(A)$, then, by virtue of external stability property [10] of set $P^n(A)$,

$$\exists t'' \in P^n(A) \;\text{ s.t. }\; g(t', t'', A) \geq 0_{(n)} \;\text{ and }\; g(t', t'', A) \neq 0_{(n)}.$$

Therefore, on account of (15), we derive

$$f_i(t^0, A_i) > f_i(t', A_i) \geq f_i(t'', A_i) \geq f_i(t'' \backslash t^0, A_i), \quad \forall i \in N_n.$$

This implies formula (14) again.

Further, let us show the sufficiency of either of the conditions (12) and (13) for stability of the problem.

If equality (12) is valid, then $\bar{P}^n(A) = \bar{S}^n(A)$. Thereby, taking into account (14), we obtain formula (2). Consequently, problem $Z^n(A)$ is stable by virtue of the Theorem 2.

If formula (13) holds, then, on account of (14) and $(S^n(A) \backslash P^n(A)) \cup \bar{S}^n(A) = \bar{P}^n(A)$, we obtain (2). Hence the problem $Z^n(A)$ is stable by the Theorem 2.

It is easy to prove the necessity of one of the conditions (12) and (13) by supposing the opposite and taking into account that formula (2) holds by virtue of the Theorem 2. □

The Corollary 3 implies the following statement.

COROLLARY 4. *Nontrivial scalar problem $Z^1(A)$ is stable for any vector $A \in \mathbb{R}^m$.*

By virtue of the equivalence of any two norms in the finite dimensional linear space (see, for example, [11]), the results of this article are valid not only for Chebyshev norm l_∞, but also for any other norm in the space of perturbing parameters $\mathbb{R}^{nm}$.

The following example shows that both conditions (12) and (13) can be violated (the problem is not stable).

EXAMPLE. *Let $n = m = 2$,*

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad T = \{t_1, t_2\}, \quad t_1 = \{e_1\}, \quad t_2 = \{e_2\}.$$

Then

$$f(t_1, A) = (1, 0), \quad f(t_2, A) = (1, 1),$$
$$P^2(A) = \{t_1\} \neq S^2(A) = T, \quad g_1(t_2, t_1 \backslash t_2, A_1) = 0. \qquad \square$$

It is easy to see, that the examples like the above can be constructed for any number of criteria $n > 2$.

REFERENCES

[1] SERGIENKO, I. V., KOZERATSKAYA, L. N. and LEBEDEVA, T. T., *Stability Investigation and Parametric Analysis of Discrete Optimization Problem*, Naukova Dumka, Kiev, 1995.

[2] SOTSKOV, YU. N., LEONTEV, V. K. and GORDEEV, E. N., *Some concepts of stability analisis in combinatorial optimization*, Discrete Appl. Math., **58**, pp. 169–190, 1995.

[3] SOTSKOV, YU. N., TANAEV, V. S. and WERNER F., *Stability radius of an optimal schedule: a survey and recent developments*, Industrial Applications of Discrete Optimization, Kluwer, **16**, pp. 72–108, 1998.

[4] Greenberg, H. G., *An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization.*, in: D. Woodruff Editor, Advances in Computational and Stochastic Optimization, Logic Programming and Heuristic Search, Kluwer Academic Publisher, Boston, pp. 97–108, 1998.

[5] Emelichev, V. A., Girlich, E., Nikulin, Yu. V. and Podkopaev, D. P., *Stability and regularization of vector problem of integer linear programming*, Optimization, **51**, pp. 645–676, 2002.

[6] Emelichev, V. A. and Leonovich, A. M., *A sensitivity measure of the Pareto set in a vector l_∞-extreme combinatorial problem*, Computer Science Jornal of Moldova, **9**, pp. 291–304, 2001.

[7] Emelichev, V. A. and Nikulin, Yu. V., *On the stability and quasi-stability of a vector lexicographic quadric boolean programming problem*, Rev. Anal. Numér. Théor. Approx., **30**, pp. 35–46, 2001.

[8] Emelichev, V. A. and Stepanishina, Yu. V., *A quasistability of the vector nonlinear trajectorial problem with Pareto principle of optimality*, Izv. Vuzov. Matematika, **12**, pp. 27–32, 2002.

[9] Emelichev, V. A. and Leonovich, A. M., *A quasistability of the vector l_∞-extreme combinatorial problem with Pareto principle of optimality*, Buletinul Acad. de St. a Republicii Moldova. Matematica, **1**, pp. 44–50, 2001.

[10] Podinovsky, V. V. and Nogin, V. D., *Pareto Optimal Solutions in Multicriteria Problems*, Nauka, Moskow, 1982.

[11] Kolmogorov, A. N. and Fomin, S. V., *Elements of Theory of Functions and Function Analysis*, Nauka, Moskow, 1972.

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