REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 32 (2003) no. 1, pp. 39–61
ictp.acad.ro/jnaat

THE SECOND DITZIAN-TOTIK MODULUS REVISITED: REFINED ESTIMATES FOR POSITIVE LINEAR OPERATORS*

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Abstract. Direct theorems for approximation by positive linear operators in terms of the second order Ditzian-Totik modulus of smoothness are proved. Special emphasis is on the magnitude of the absolute constants. New results are obtained for Bernstein operators, for piecewise linear interpolation, for general Bernstein-Stancu operators and for those of Gavrea.

MSC 2000. 41A10, 41A15, 41A36.

Keywords. Bernstein operator, Ditzian-Totik moduli of smoothness, best constants, piecewise linear interpolation, Bernstein-Stancu operator, Gavrea operator.

1. INTRODUCTION

At the beginning of the 80's it had become clear that for characterizing those functions $f \in C[0,1]$ for which the quantities $\|f - B_n f\|_\infty$ vanish at a given speed, moduli of smoothness should be used which are based on differences Δ_u^2 in which the step u is allowed to depend upon the position of x in the interval $[0,1]$. Here B_n denotes the Bernstein operator given by

$$B_n(f,x) = \sum_{k=1}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad n \in \mathbb{N},$$

and a corresponding statement is also true for similar operators.

At the forefront of the development were authors such as Z. Ditzian, K. Ivanov, V. Totik and Xin-long Zhou and several others not mentioned here. Many references to their work can be found in two bibliographies in which the work on Bernstein-type operators up to the middle of the 80's (see [17], [18]) was compiled. What is missing in the latter, though, is an entry with the important master thesis of Xin-long Zhou [37] finished already in 1981, which is, however, only available in handwritten Chinese.

The continuing work of the authors mentioned explicitely in the above eventually culminated in, among other articles, a book by Ditzian and Totik [9],

*The work of the second author has been partially supported by DAAD-Grant A/98/03004.

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joint work of Ditzian and Zhou (see, e.g., [10]), an important article by Ditzian and Ivanov [8], a paper by Totik [36], and a significant contribution of Knoop and Zhou [23], [24]. A somewhat partial, but nonetheless streamlined, account of what had been achieved up to 1993, say, is given in the book of DeVore and Lorentz [5].

Knoop and Zhou proved the following strong result: There are constants c_1 and c_2 such that for $n \in \mathbb{N}$ one has

$$c_1 \cdot \omega_2^{\varphi}\big(f; \tfrac{1}{\sqrt{n}}\big) \le \|B_n f - f\|_\infty \le c_2 \cdot \omega_2^{\varphi}\big(f; \tfrac{1}{\sqrt{n}}\big). \tag{1.1}$$

The left inequality is usually called a "strong converse inequality", while the right one can be denoted as a "Jackson inequality (in terms of ω_2^{φ})".

To our knowledge, the Jackson-type inequality was first explicitly presented in a paper by Xin-long Zhou [38], an article submitted in 1982 (see also Ditzian [6], [7]). This historical fact *might* justify calling the modulus in question the Ditzian-Totik-Zhou modulus, but we will refrain to do so. In the same year (1982) a paper by Ivanov [19] was published in which an analogous direct theorem for approximation by Bernstein operators was given, but in terms of τ-moduli. This was done independently of Xin-long Zhou, and only 9 years later it was again Ivanov who established the equivalence between the two types of moduli in his treatise [21]. An email exchange with Professor Berens of October 1996 revealed that for him it would be worthwhile to look for the best value of the constant c_2. The present note intends to present, among other things, a first modest step into this direction.

Before proceeding further, the following need to be mentioned for completeness.

The above quantity $\omega_2^{\varphi}(f;\cdot)$ is defined by

$$\omega_2^{\varphi}(f;t) = \sup_{0\le h\le t} \|\Delta^2_{h\varphi} f\|_\infty, \quad \varphi(x) = \sqrt{x(1-x)},$$

where

$$\Delta^2_{h\varphi} f(x) := \begin{cases} f\big(x - h\varphi(x)\big) - 2f(x) + f\big(x + h\varphi(x)\big), \\ \qquad\qquad\qquad \text{if } [x - h\varphi(x), x + h\varphi(x)] \subseteq [0,1]; \\ 0, \quad \text{otherwise.} \end{cases}$$

The upper estimate in (1.1) was proved using the K-functional K_2^{φ} given, for $f \in C[0,1]$ and $t \ge 0$, by

$$K_2^{\varphi}(f, t^2) := \inf_g \big\{\|f - g\|_\infty + t^2 \cdot \|\varphi^2 \cdot g''\|_\infty\big\},$$

where the infimum is taken over all g such that $g' \in AC_{\mathrm{loc}}[0,1]$ (i.e., g' is absolutely continuous in $[a,b]$ for every a, b satisfying $0 < a < b < 1$) and $\|\varphi^2 g''\|_\infty < \infty$.

Throughout this note we shall use this definition of $K_2^\varphi(f,t^2)$ introduced by Ditzian and Totik (see (2.11) in [9]). Let $W_{2,\infty}^\varphi$ be the corresponding weighted Sobolev space with weight function φ^2 consisting of all such functions g.

Remark 1.1. There are various different definitions of $K_2^\varphi(f,t^2)$ figuring in the literature. We mention the following ones:

$$\begin{aligned} K_2^\varphi(f,t^2) &= \inf\left\{\|f-g\|_\infty + t^2\cdot\|\varphi^2 g''\|_\infty : g' \in AC[0,1]\right\} \\ K_2^\varphi(f,t^2) &= \inf\left\{\|f-g\|_\infty + t^2\cdot\|\varphi^2 g''\|_\infty : g \in W_{2,\infty}[0,1]\right\} \\ K_2^\varphi(f,t^2) &= \inf\left\{\|f-g\|_\infty + t^2\cdot\|\varphi^2 g''\|_\infty : g \in C^2[0,1]\right\} \end{aligned}$$

It appears to be of interest to investigate the differences between these functionals, and also the ones between these and what we are using here, namely

$$K_2^\varphi(f,t^2) = \inf\left\{\|f-g\|_\infty + t^2\cdot\|\varphi^2 g''\|_\infty : g' \in AC_{\mathrm{loc}}[0,1]\right\}. \qquad \square$$

No *concise* description of the relationships is known to us.

In order to arrive at an inequality in terms of ω_2^φ, the equivalence

$$c_3\omega_2^\varphi(f,t) \le K_2^\varphi(f,t^2) \le c_4\omega_2^\varphi(f,t), \quad 0 \le t \le t_0,$$

was used (see [9, Theorem 2.1.1], [5, formula (7.5)]). Since the number t_0 will be relevant to us, we will specify it later.

As far as we know, the problem to find the best (or at least some) constants c_3, c_4 for which the above equivalence holds, remains still open. The same statement can be made about the constants c_1 and c_2 in (1.1). The latter state of the art in regard to ω_2^φ is in sharp contrast/backlog to what is known/has been claimed with respect to two related estimates:

(1) In 1998 Păltănea announced on two occasions that the uniform Brudnyĭ-type estimate for the classical Bernstein operators (see [2]) reads

$$\|B_n f - f\|_\infty \le 1\cdot\omega_2\Big(f;\tfrac{1}{\sqrt{n}}\Big), \quad f \in C[0,1],\ n \in \mathbb{N},$$

where the constant 1 is best possible (see [29] and [15] for details).

(2) The pointwise Cao-type inequality for Bernstein operators (see [4]), namely

$$\big|B_n(f;x) - f(x)\big| \le c_5\cdot\omega_2\Big(f;\sqrt{\tfrac{x(1-x)}{n}}\Big), \quad f \in C[0,1],\ n \in \mathbb{N},\ x \in [0,1],$$

can be made more precise by proving that the best possible value of c_5 is ≤ 1.5. This is one consequence of a general result of Păltănea [28, Theorem 2.1] which we cite as

Theorem A. *If $L : C[0,1] \to B[0,1]$, the space of bounded functions, is a positive linear operator reproducing linear functions, then for $f \in C[0,1]$, $x \in [0,1]$ and each $0 < h \le \frac{1}{2}$, the following holds:*

$$\big|L(f;x) - f(x)\big| \le \left[1 + \tfrac{1}{2}\cdot h^{-2}\cdot L\big((e_1 - x)^2; x\big)\right]\cdot\omega_2(f;h).$$

Here $e_1(t) = t$, $t \in [0,1]$, *and* $\omega_2(f;\cdot)$ *denotes the* (*classical*) *second order modulus of* f.

It is the aim of the present note to derive an analogy of Păltănea's result (see also [16]) for estimates in terms of ω_2^{φ}, to give a more precise upper bound for approximation by Bernstein operators and to apply the new general inequality to further positive linear operators reproducing linear functions. As a byproduct we will also obtain a more precise form of the equivalence between K_2^{φ} and ω_2^{φ}.

Remark 1.2. During the preparation of this paper the authors strived to obtain more information on the historical roots of $\omega_2^{\varphi}(f;\cdot)$ and $K_2^{\varphi}(f;\cdot)$. This is not an easy task since, on average, mathematicians hardly care about the history and the global aspect of their subject. For the time being we decided to just add references [11] and [31] to the ones give in the book of Ditzian and Totik. Both were taken from Ivanov's paper [20]. □

2. AN AUXILIARY RESULT AND SOME REMARKS ON ω_2^{φ}

The following inequality will be indispensable for our later considerations.

Lemma 2.1 (Burkill [3, Lemma 5.2]). *For a compact interval* $[\alpha,\beta]$ *and* $f \in C[\alpha,\beta]$, *let* ℓ *denote the linear function interpolating* f *at* α *and* β. *Then*

$$|f(x)-\ell(x)| \le \omega_2\big(f;\tfrac{\beta-\alpha}{2}\big), \quad \textit{for all } x \in [\alpha,\beta].$$

It is clear from the definition of the classical ω_2 that $\omega_2(f;h) = \omega_2(f;\frac{\beta-\alpha}{2})$ for $h \ge \frac{\beta-\alpha}{2}$. The situation is similar for ω_2^{φ} (we focus on $[\alpha,\beta]=[0,1]$ again). So, as before, $\varphi(x) = \sqrt{x(1-x)}$.

Let $H \ge 0$. Then differences of the form

$$f\big(x - H\varphi(x)\big) - 2f(x) + f\big(x + H\varphi(x)\big), \quad x \in [0,1],$$

contribute to an actual value of $\omega_2^{\varphi}(f,\cdot)$ only if

$$0 \le x - H\cdot\varphi(x) \quad \text{and} \quad x + H\cdot\varphi(x) \le 1.$$

As can be seen by inspection, this is the case if and only if

$$H^2 \le M(x) := \min\left\{\tfrac{1-x}{x}, \tfrac{x}{1-x}\right\} \le 1,$$

or

$$0 \le H \le \min\left\{\sqrt{\tfrac{1-x}{x}}, \sqrt{\tfrac{x}{1-x}}\right\} \le 1.$$

That is, for $t > 1$ one has

$$\omega_2^{\varphi}(f;t) = \omega_2^{\varphi}(f;1). \tag{2.1}$$

The functional $K_2^{\varphi}(f,t^2)$, in contrary, is well-defined for any $t \ge 0$. But due to (2.1), in the sequel it will be sufficient to deal with inequalities of the type

$$c_3 \cdot \omega_2^{\varphi}(f;t) \le K_2^{\varphi}(f,t^2) \le c_4 \cdot \omega_2^{\varphi}(f;t)$$

for $0 \le t \le 1$ only. Also, sometimes it will be advantageous to deal with the analogous inequality

$$c_3 \cdot \omega_2^{\varphi}(f;t) \le K_2^{\varphi}(f,t^2) \le c_4(\gamma) \cdot \omega_2^{\varphi}(f;t),$$

where t is now restricted to the range $0 \le t \le \gamma$. In certain instances below, the latter observation will become relevant for our purposes.

Several papers appeared in the past in which the question was dealt with of how to bridge the gap between the use of $\omega_2(f;\cdot) = \omega_2^1(f;\cdot) = \omega_2^{\varphi^0}(f;\cdot)$ and that of $\omega_2^{\varphi}(f;\cdot) = \omega_2^{\varphi^1}(f;\cdot)$ in estimates for (positive) linear operators. See [15] and the references cited there for more information about the historical background.

The relevant moduli $\omega_2^{\varphi^\lambda}$, $0 \le \lambda \le 1$, will also figure below. Their definition is in complete analogy to those of $\omega_2(f;\cdot)$ and $\omega_2^{\varphi}(f;\cdot)$, namely

$$\omega_2^{\varphi^\lambda}(f;t) := \sup_{0 \le h \le t} \|\Delta^2_{h\varphi^\lambda} f\|_\infty,$$

where

$$\Delta^2_{h\varphi^\lambda} f(x) := \begin{cases} f\big(x - h\varphi^\lambda(x)\big) - 2f(x) + f\big(x + h\varphi^\lambda(x)\big), \\ \qquad\qquad\qquad \text{if } [x - h\varphi^\lambda(x), x + h\varphi^\lambda(x)] \subseteq [0,1]; \\ 0, \quad \text{otherwise.} \end{cases}$$

For $\lambda = 0$, this is the definition of $\omega_2(f;\cdot) = \omega_2^1(f;\cdot)$. For $\lambda = 1$, we get that of $\omega_2^{\varphi}(f;\cdot)$. As was observed earlier, they become constant at different values of the second parameter t:

$$\begin{aligned} \omega_2(f;t) &= \omega_2(f;\tfrac{1}{2}), \quad \text{for } t \ge \tfrac{1}{2}, \\ \omega_2^{\varphi}(f;t) &= \omega_2(f;1), \quad \text{for } t \ge 1. \end{aligned}$$

Considerations similar to those for the case of ω_2^{φ} yield that, for $0 \le \lambda \le 1$,

$$\omega_2^{\varphi^\lambda}(f;t) = \omega_2^{\varphi^\lambda}\big(f;(\tfrac{1}{2})^{1-\lambda}\big), \quad \text{for } t \ge (\tfrac{1}{2})^{1-\lambda},$$

which generalizes the previous two equalities.

Associated with $\omega_2^{\varphi^\lambda}$ is the functional

$$K_2^{\varphi^\lambda}(f;t^2) := \inf\big\{\|f-g\|_\infty + t^2 \cdot \|\varphi^{2\lambda} \cdot g''\|_\infty\big\}, \quad t \ge 0, \tag{2.2}$$

where the infimum is taken over all g such that $g' \in AC_{\mathrm{loc}}[0,1]$ and $\|\varphi^{2\lambda} \cdot g''\|_\infty < \infty$.

When considering inequalities of the type

$$K_2^{\varphi^\lambda}(f;t^2) \le c_4(\lambda,t_0) \cdot \omega_2^{\varphi^\lambda}(f;t), \quad 0 \le t \le t_0,$$

it is therefore natural to consider these for $t \leq (\frac{1}{2})^{1-\lambda}$ first and to deal with values of $T \geq (\frac{1}{2})^{1-\lambda} =: t_{\max}^{(\lambda)}$ only afterwards. If such T is given, then

$$\begin{aligned} K_2^{\varphi^\lambda}(f;T^2) &= \inf\left\{\|f-g\|_\infty + \left(\tfrac{T}{t_{\max}^{(\lambda)}}\right)^2 \cdot \left(t_{\max}^{(\lambda)}\right)^2 \cdot \|\varphi^{2\lambda}\cdot g''\|_\infty\right\} \\ &\leq \left(\tfrac{T}{t_{\max}^{(\lambda)}}\right)^2 \cdot c_4(\lambda, t_{\max}^{(\lambda)}) \cdot \omega_2^{\varphi^\lambda}(f; t_{\max}^{(\lambda)}). \end{aligned}$$

Hence for all $t \geq 0$ we have

$$K_2^{\varphi^\lambda}(f;t^2) \leq \max\left\{1, \left(\tfrac{t}{t_{\max}^{(\lambda)}}\right)^2\right\} \cdot c_4(\lambda, t_{\max}^{(\lambda)}) \cdot \omega_2^{\varphi^\lambda}(f;t).$$

3. MAIN RESULTS

We recall that it was observed in [8] and [12] that

$$\|f - B_n f\|_\infty \leq 2 \cdot K_2^{\varphi}(f, \tfrac{1}{n}).$$

The method from there will also be applied in the first step of the proof of our main result. In order to formulate it, we need to introduce the following sequence $d(m)$.

For m a natural number we set

$$d := d(m) = \tfrac{\sqrt{m^4+m^2+1}-1}{\sqrt{m^4+m^2+1}+m^2}.$$

It is easy to compute

$$d(1) \approx 0.268,\ \ d(2) \approx 0.4174,\ \ d(3) \approx 0.4606,\ \ d(10) \approx 0.4962,\ \ \lim_{m\to\infty} d(m) = \tfrac{1}{2},$$

and to note that the sequence $(d(m))_{m\geq 1}$ is strictly increasing. As a consequence,

$$\left(\tfrac{1}{m\cdot d(m)}\right)_{m\geq 1}$$

strictly decreases to zero.

The significance of $d(m)$ will become clear in the course of the proof of the following

THEOREM 3.1. *If* $L : C[0,1] \to C[0,1]$ *is a linear positive operator which preserves linear functions,* $h \in \left[\frac{\sqrt{2}}{md(m)}, \frac{\sqrt{2}}{(m-1)d(m-1)}\right)$, $m \geq 2$ *a natural number, then*

$$\left|Lf(x) - f(x)\right| \leq \left[2 + \left(\tfrac{m}{m-1}\right)^2 \tfrac{48}{d^2(m-1)} \tfrac{L((e_1-x)^2;x)}{h^2\varphi^2(x)}\right]\omega_2^{\varphi}(f,h).$$

Sketch of proof of Theorem 3.1. As was indicated above, the well-known smoothing technique will be applied. As in the proof of Theorem 1 in [12] and Theorem 8.1 in [8] it follows that

$$\begin{aligned} \left|Lf(x) - f(x)\right| &\leq \left|L(f-g)(x)\right| + \left|f(x)-g(x)\right| + \left|Lg(x) - g(x)\right| \\ &\leq 2\left\|f-g\right\|_\infty + \tfrac{L((e_1-x)^2;x)}{\varphi^2(x)}\|\varphi^2 g''\|_\infty \end{aligned}$$

for arbitrary $g \in W^{\varphi}_{2,\infty}$, and taking into account that $\|L\| = 1$. To choose an appropriate function g we follow some ideas from [16]. Like there, the function g will be constructed in a two stage process, will be a spline in $W^{\varphi}_{2,\infty}$ (with non-equidistant knots), and will satisfy here inequalities of the types

$$\|f - g\|_\infty \le \omega_2^\varphi\left(f, \tfrac{\sqrt{2}}{m\cdot d(m)}\right)$$

and

$$\|\varphi^2 g''\|_\infty \le 24m^2 \cdot \omega_2^\varphi\left(\tfrac{\sqrt{2}}{m\cdot d(m)}\right),$$

where the natural number $m \ge 2$ is related to h as described in the theorem. Clearly, g will depend on h (via m) and on the function φ.

As mentioned earlier, g will be constructed in two steps. The first step will be to define a piecewise linear continuous function based upon an appropriate sequence of non-equidistant knots. The construction and properties of these are given in Lemma 3.2. In Lemma 3.3 we will eventually construct the spline g and prove its quantitative properties listed above. □

We now turn to

LEMMA 3.2. *For $m \ge 2$, let*

$$y_0 = 0, \quad y_1 = \tfrac{1}{m^2+1}, \quad y_2 = y_1 + \tfrac{\varphi(y_1)}{m}, \ldots, y_{k+1} = y_k + \tfrac{\varphi(y_k)}{m}, \quad k = 1, 2, \ldots, M-1,$$

where M is the biggest number, such that

$$z_M = y_M + \tfrac{\varphi(y_M)}{M} < \tfrac{1}{2},$$

$$z_M + \tfrac{\varphi(z_M)}{M} < \tfrac{1}{2}. \tag{3.1}$$

We set

$$y_{M+1} = \tfrac{1}{2},$$

and, symmetrically,

$$y_{M+2} = 1 - y_M, \quad y_{M+3} = 1 - y_{M-1}, \ldots, y_{2M+1} = 1 - y_1, \quad y_{2M+2} = 1.$$

Then, for $k = 1, 2, \ldots, 2M$ and $x \in [y_k, y_{k+1}]$, we have

$$d(m)(y_{k+1} - y_k) \le \tfrac{\varphi(x)}{m} \le \sqrt{2}(y_{k+1} - y_k). \tag{3.2}$$

Proof. By symmetry it is enough to consider $k = 1, 2, \ldots, M$. First we consider $1 \le k \le M-1$, and then the case $k = M$ separately. The function $\varphi(x)$ is strictly increasing for $0 \le x < \frac{1}{2}$. Hence, for $x \in [y_k, y_{k+1}]$,

$$y_{k+1} - y_k = \tfrac{\varphi(y_k)}{m} \le \tfrac{\varphi(x)}{m} \le \tfrac{\varphi(y_{k+1})}{m}.$$

We claim that

$$
\begin{array}{llrcl}
 & & \varphi^2(y_{k+1}) & \le & 2\varphi^2(y_k) \\
 & \Longleftrightarrow & y_{k+1}(1-y_{k+1}) & \le & 2y_k(1-y_k) \\
(3.3) & \Longleftrightarrow & \left(y_k+\frac{\varphi(y_k)}{m}\right)\left(1-y_k-\frac{\varphi(y_k)}{m}\right) & \le & 2y_k(1-y_k) \\
 & \Longleftarrow & \frac{\varphi(y_k)}{m}(1-2y_k) & \le & y_k(1-y_k)=\varphi^2(y_k) \\
 & \Longleftrightarrow & \frac{1-2y_k}{m} & \le & \varphi(y_k).
\end{array}
$$

The latter inequality is fulfilled for

$$y_k \in \left[\frac{1}{2}\left(1-\frac{m}{\sqrt{m^2+4}}\right), \frac{1}{2}\left(1+\frac{m}{\sqrt{m^2+4}}\right)\right].$$

So the question remains if all y_k satisfy the latter inequality.

After simple calculation we obtain

$$y_k \ge y_1 = \frac{1}{1+m^2} > \frac{1}{2}\left(1-\frac{m}{\sqrt{m^2+4}}\right),$$

and so (3.3) is proved.

Now we consider the case $k=M$, so let $x \in [y_M, \frac{1}{2}]$. It only remains to prove that

$$d(m)\left(\frac{1}{2}-y_M\right) \le \frac{\varphi(x)}{m} \le \sqrt{2}\left(\frac{1}{2}-y_M\right).$$

Obviously

$$\frac{\varphi(y_M)}{m} \le \frac{\varphi(x)}{m} \le \frac{1}{2m}.$$

Therefore it is enough to establish

$$\frac{1}{2m} \le \sqrt{2}\left(\frac{1}{2}-y_M\right) \tag{3.4}$$

and

$$d(m)\left(\frac{1}{2}-y_M\right) \le \frac{\varphi(y_M)}{m}. \tag{3.5}$$

The inequalities (3.4)–(3.5) are satisfied for

$$y_M \in \left[\frac{1}{2}\left(1-\frac{1}{\sqrt{m^2d^2(m)+1}}\right), \frac{1}{2}\left(1-\frac{1}{\sqrt{2m^2}}\right)\right]. \tag{3.6}$$

The inequalities (3.1) and (3.2) are satisfied for

$$y_M \in \left[\frac{1}{2}\left(1-\frac{m^2+\sqrt{m^4+m^2+1}}{(m^2+1)\sqrt{m^2+1}}\right), \frac{1}{2}\left(1-\frac{1}{\sqrt{m^2+1}}\right)\right]. \tag{3.7}$$

It is not difficult to verify that the interval in (3.6) contains that in (3.7). With this the proof of Lemma 3.2 is completed. □

Having proven Lemma 3.2, we are now in the state to construct the desired function g using an intermediate continuous polygonal spline $S_1(f)$, $f \in C[0,1]$.

At the points $y_k,\ k=0,\dots,2M+2$, the function f is interpolated by the polygonal spline $S_1(f)$. Each point $(y_k, S_1(y_k)),\ k=1,\dots,2M+1$, we associate with two other points $(a_k, S_1(a_k)),\ (b_k, S_1(b_k))$, such that

$$a_1 = \tfrac{y_1}{2}, \quad b_1 - y_1 = y_1 - a_1,$$

and

$$a_k = \tfrac{y_k + y_{k-1}}{2}, \quad b_k - y_k = y_k - a_k, \quad k = 1, 2, \dots, M+1.$$

We define the points $(a_k, S_1(a_k)),\ (b_k, S_1(b_k)),\ k = M+2,\dots,2M+1$, by symmetry according to $\frac{1}{2}$. Using the definition of the knots $\{y_k\}$ it is clear that

$$y_1 = \tfrac{1}{1+m^2}, \quad y_2 = \tfrac{2}{1+m^2}, \quad a_2 = b_1,$$

$$a_k < y_k < b_k < a_{k+1} < y_{k+1} < b_{k+1}, \quad 2 \le k \le 2M, \quad y_k = \tfrac{a_k + b_k}{2},$$

$$b_k - a_k = 2(y_k - a_k) = y_k - y_{k-1} = \tfrac{\varphi(y_{k-1})}{m}.$$

We are ready to define the function g.

For $x \in [0, a_1] \cup [b_{2M+1}, 1]$, we set $g(x) \equiv S_1(f)(x)$.

For $x \in [a_k, b_k]$, $k = 1,\dots,2M+1$, $g(x)$ is the 2nd degree Bernstein polynomial over the interval $[a_k, b_k]$, determined by the ordinates $S_1(a_k),\ f(y_k),\ S_1(b_k)$.

For $x \in [b_k, a_{k+1}]$, $k = 1,\dots,2M$, we set $g(x) \equiv S_1(f)(x)$.

Thus, $g(x)$ is uniquely determined by the interpolation conditions, is C^1-continuous and can be composed of "Bernstein parabolas" by the well-known control point construction, and of some straight line segments.

Let $k = 1,\dots,M$. From the convex-hull property of Bernstein operator and the considerations in [16] (see pp. 28–29) for $x \in [a_k, b_k]$, it follows that

$$\big|(f-g)(x)\big| \le \omega_2\big(f; \tfrac{y_{k+1} - y_{k-1}}{2}\big)_{[y_{k-1}, y_{k+1}]} \le \omega_2\big(f; y_{k+1} - y_k\big)_{[y_{k-1}, y_{k+1}]}. \tag{3.8}$$

Here $\omega_2(f, \cdot)_{[c,d]}$ denotes the classical second order modulus of f restricted to the interval $[c, d]$. We consider two cases:

Case I: $x \in [y_k, b_k]$, $k = 1,\dots,M$.

From (3.2) and (3.8) we get

$$\begin{aligned}\big|(f-g)(x)\big| &\le \sup\Big\{\big|f(x+h) - 2f(x) + f(x-h)\big|, \quad x \in [y_k, b_k],\\ &\qquad x \pm h \in [y_{k-1}, y_{k+1}],\ |h| \le y_{k+1} - y_k \le \tfrac{\varphi(x)}{md(m)}\Big\}\\ &\le \sup\Big\{\big|f(x+h) - 2f(x) + f(x-h)\big|, \quad x \in [y_k, b_k],\\ &\qquad x \pm h \in [y_{k-1}, y_{k+1}],\ |h| \le \tfrac{\varphi(x)}{md(m)}\Big\}.\end{aligned} \tag{3.9}$$

Hence

$$\big|f(x) - g(x)\big| \le \omega_2^{\varphi}\big(f, \tfrac{1}{md(m)}\big). \tag{3.10}$$

Case II: $x \in [a_k, y_k]$, $k = 1,\dots,M$.

Following (3.2), it is easy to observe that

$$y_{k+1}-y_k=\tfrac{\varphi(y_k)}{m}\le\sqrt{2}(y_k-y_{k-1})\le\tfrac{\sqrt{2}\varphi(x)}{md(m)},$$

for $x\in[a_k,y_k]\subset[y_{k-1},y_k]$.

The last inequality and (3.10) yield, for $x\in[a_k,b_k]$,

$$\big|(f-g)(x)\big|\le\omega_2^{\varphi}\big(f,\tfrac{\sqrt{2}}{md(m)}\big).$$

Cases I and II from above cover the cases $x\in[a_1,b_M]$.

Let $x\in[a_{M+1},\frac{1}{2}]$. Similarly we obtain

$$\big|(f-g)(x)\big|\le\omega_2\big(f;y_{M+1}-y_M\big)\le\omega_2^{\varphi}\big(f,\tfrac{1}{md(m)}\big).$$

By the definition of the function $g(x)$, for $x\in[b_k,a_{k+1}]$, $k=1,\ldots,M$, $g(x)\equiv S_1(f)(x)$ and in this case

$$\big|(f-g)(x)\big|\le\omega_2\big(f;\tfrac{y_{k+1}-y_k}{2}\big)\le\omega_2^{\varphi}\big(f,\tfrac{1}{2md(m)}\big).$$

From (3.8)–(3.10) we have ($\|\cdot\|_{C[c,d]}$ denoting the sup norm over $[c,d]$)

$$\|f-g\|_{C[a_1,\frac{1}{2}]}\le\omega_2^{\varphi}\big(f,\tfrac{\sqrt{2}}{md(m)}\big). \tag{3.11}$$

It remains to consider $x\in[0,a_1]$. Over the interval $[0,a_1]$, using the same arguments as above we get

$$\begin{aligned}\big|(f-g)(x)\big| \;\le\; \sup\big\{&\big|f(x+h)-2f(x)+f(x-h)\big|,\\ &x,x\pm h\in[0,y_1],\ |h|<a_1\big\}.\end{aligned} \tag{3.12}$$

From $x-h\ge 0$, it follows

$$h\le x\le a_1=\tfrac{1}{2(1+m^2)}. \tag{3.13}$$

But $x<\varphi(x)/m$ holds if and only if $x<(1+m^2)^{-1}$, and the latter is true due to (3.13). In other words, we have (3.12) for $h<\varphi(x)/m$. Hence

$$\|f-g\|_{C[0,a_1]}\le\omega_2^{\varphi}\big(f,\tfrac{1}{m}\big). \tag{3.14}$$

By symmetry, the cases $x\in[b_{2M+1},1]$ and $x\in[\frac{1}{2},b_{2M+1}]$ are analogous to the ones previously considered. Combining (3.11)–(3.14), we arrive at

$$\|f-g\|_{\infty}\le\omega_2^{\varphi}\big(f,\tfrac{\sqrt{2}}{md(m)}\big). \tag{3.15}$$

We consider g'' next. If $x\in[0,a_1]\cup[b_k,a_{k+1}]\cup[b_{2M+1},1]$, $g(x)$ is a linear function and thus $g''(x)=0$.

Let $x \in [a_k, b_k]$, $k = 1, \ldots, 2M+1$. We use the fact, that

$$
\begin{aligned}
&|g''(x)| = \\
&= \tfrac{2}{(b_k-a_k)^2}\big|g(b_k) - 2g(y_k) + g(a_k)\big| \\
&\le \tfrac{2}{(b_k-a_k)^2}\Big[\big|f(b_k) - 2f(y_k) + f(a_k)\big| + \big|g(b_k) - f(b_k)\big| + \big|g(a_k) - f(a_k)\big|\Big] \\
&\le \tfrac{2}{(b_k-a_k)^2}\Big[\omega_2\big(f; \tfrac{b_k-a_k}{2}\big) + 2\omega_2^{\varphi}\big(f, \tfrac{\sqrt{2}}{md(m)}\big)\Big] \\
&= \tfrac{2}{(b_k-a_k)^2}\Big[\omega_2\big(f; \tfrac{y_k-y_{k-1}}{2}\big) + 2\omega_2^{\varphi}\big(f, \tfrac{\sqrt{2}}{md(m)}\big)\Big]
\end{aligned}
$$

(3.16)

$$\le \tfrac{6}{(y_k-y_{k-1})^2}\omega_2^{\varphi}\big(f, \tfrac{\sqrt{2}}{md(m)}\big).$$

In order to arrive at the last inequality we have applied (3.2) and (3.15).

For $x \in [a_k, y_k]$, from (3.16) and (3.2) we get

$$|\varphi^2(x)g''(x)| \le 12m^2\omega_2^{\varphi}\big(f, \tfrac{\sqrt{2}}{md(m)}\big).$$

For $x \in [y_k, b_k]$, analogously,

$$
\begin{aligned}
\big|\varphi^2(x)g''(x)\big| &\le 12m^2\tfrac{(y_{k+1}-y_k)^2}{(y_k-y_{k-1})^2}\omega_2^{\varphi}\big(f, \tfrac{\sqrt{2}}{md(m)}\big) \\
&= 12m^2\left(\tfrac{\varphi(y_k)/m}{y_k-y_{k-1}}\right)^2\omega_2^{\varphi}\big(f, \tfrac{\sqrt{2}}{md(m)}\big) \\
&\le 24m^2\omega_2^{\varphi}\big(f, \tfrac{\sqrt{2}}{md(m)}\big), \quad \text{by (3.2), and since } y_k \in [y_{k-1}, y_k].
\end{aligned}
$$

Hence,

$$\|\varphi^2 g''\|_\infty \le 24m^2\omega_2^{\varphi}\big(f, \tfrac{\sqrt{2}}{md(m)}\big).$$

By the previous considerations we have shown the validity of

LEMMA 3.3. *Let $g = g_{m,\varphi}$ be the above quadratic C^1-spline based upon the knot sequence*

$$\overline{y} = \big(0 = y_0 < y_1 < \ldots < y_M < y_{M+1} = \tfrac{1}{2} < y_{M+2} < \ldots < y_{M+2} = 1\big).$$

Then

(i) $$\|f-g\|_\infty \le \omega_2^{\varphi}\big(f; \tfrac{\sqrt{2}}{md(m)}\big), \quad \textit{and}$$

(ii) $$\|\varphi^2 g''\|_\infty \le 24m^2 \cdot \omega_2^{\varphi}\big(f; \tfrac{\sqrt{2}}{md(m)}\big).$$

Here $m \ge 2$ is any natural number, and $d(m)$ denotes the sequence defined prior to Theorem 3.1.

REMARK 3.1. (i) We recall our earlier observation that

$$\omega_2^{\varphi}(f; t) = \omega_2^{\varphi}(f; 1), \quad \text{for } t \ge 1.$$

This means that Lemma 3.3 can be formulated also in the following way:

$$\text{(i')} \qquad \|f-g\|_\infty \le \omega_2^\varphi\Big(f; \min\big\{\tfrac{\sqrt{2}}{md(m)}, 1\big\}\Big),$$

$$\text{(ii')} \qquad \|\varphi^2 g''\|_\infty \le 24m^2 \cdot \omega_2^\varphi\Big(f; \min\big\{\tfrac{\sqrt{2}}{md(m)}, 1\big\}\Big).$$

In order to cover the range $0 < t \le 1$, it will be sufficient to consider $m \ge m_0$, where

$$\text{l.s.} = \tfrac{\sqrt{2}}{(m_0+1)d(m_0+1)} \le 1 < \tfrac{\sqrt{2}}{m_0 d(m_0)} = \text{r.s.}$$

This inequality holds for $m_0 = 3$, in which case we have

$$\text{l.s.} = \frac{\sqrt{2}}{4 \cdot \frac{\sqrt{256+16+1}-1}{\sqrt{256+16+1}+16}} < 1 < 1.0234574 \approx \text{r.s.}$$

(ii) The observation made in (i) is sometimes useful to reduce the magnitude of constants. We postpone a confirmation of this until after the proofs of Theorems 3.1 and 3.4. □

We are now ready to finalize the

Proof of Theorem 3.1. We recall that

$$\big|Lf(x) - f(x)\big| \le 2 \cdot \|f-g\|_\infty + \tfrac{L((e_1-x)^2;x)}{\varphi^2(x)} \cdot \|\varphi^2 g''\|_\infty$$

for arbitrary $g \in W_{2,\infty}^\varphi$, and substitute for g the function $g_{m,\varphi}$ from Lemma 3.3. This furnishes, for any $m \ge 2$,

$$\begin{aligned} \big|Lf(x) - f(x)\big| &\le \Big[2 + 24m^2 \cdot \tfrac{L((e_1-x)^2;x)}{\varphi^2(x)}\Big] \cdot \omega_2^\varphi\big(f; \tfrac{\sqrt{2}}{md(m)}\big) \\ &= \Big[2 + 24m^2 \cdot h^2 \cdot \tfrac{L((e_1-x)^2;x)}{h^2 \cdot \varphi^2(x)}\Big] \cdot \omega_2^\varphi\big(f; \tfrac{\sqrt{2}}{md(m)}\big), \end{aligned}$$

where $h > 0$ is arbitrary. For every $0 < h < \sqrt{2}/d(1)$, there exists $m \ge 2$ such that

$$\tfrac{\sqrt{2}}{m \cdot d(m)} \le h < \tfrac{\sqrt{2}}{(m-1) \cdot d(m-1)}.$$

Substituting these bounds for h shows the validity of Theorem 3.1. □

As was mentioned in the introduction, we also have the following

Theorem 3.4. *For the K-functional K_2^φ defined above, $m \ge 2$, and $h \in \big[\tfrac{\sqrt{2}}{m \cdot d(m)}, \tfrac{\sqrt{2}}{(m-1) \cdot d(m-1)}\big)$, the following inequalities hold for any $f \in C[0,1]$:*

$$\tfrac{1}{16} \cdot \omega_2^\varphi(f;h) \le K_2^\varphi(f;h^2) \le \Big[1 + \big(\tfrac{m}{m-1}\big)^2 \cdot \tfrac{48}{d^2(m-1)}\Big] \cdot \omega_2^\varphi(f;h).$$

Proof. In order to derive the lower bound we follow the proof of Theorem 6.1 in [5] (see pp. 187–188). This shows that it is possible to take $\frac{1}{16}$ as a possible value for the constant c_3.

For getting the upper bound, we choose m as determined by h and use the function $g_{m,\varphi}$ in order to find

$$\begin{aligned} K_2^{\varphi}(f;h^2) &\le \|f-g_{m,\varphi}\|_\infty + h^2\cdot\|\varphi^2\cdot g''_{m,\varphi}\|_\infty \\ &\le \omega_2^{\varphi}\big(f;\tfrac{\sqrt{2}}{m\cdot d(m)}\big) + \tfrac{48m^2}{(m-1)^2\cdot d^2(m-1)}\cdot\omega_2^{\varphi}\big(f;\tfrac{\sqrt{2}}{m\cdot d(m)}\big) \\ &\le \Big(1+\big(\tfrac{m}{m-1}\big)^2\cdot\tfrac{48}{d^2(m-1)}\Big)\cdot\omega_2^{\varphi}(f;h). \end{aligned}$$ □

Corollary 3.5. (i) *For $h\in\big(0,\frac{\sqrt{2}}{d(1)}\big)\approx(0,5.2769)$ we have*

$$\tfrac{1}{16}\cdot\omega_2^{\varphi}(f;h)\le K_2^{\varphi}(f;h^2)\le 2675\cdot\omega_2^{\varphi}(f;h).$$

(ii) *Defining $C:\big(0,\frac{\sqrt{2}}{d(1)}\big)\to\mathbb{R}$ by*

$$C(h):=1+\big(\tfrac{m}{m-1}\big)^2\cdot\tfrac{48}{d^2(m-1)},\quad \text{if } h\in\big[\tfrac{\sqrt{2}}{m\cdot d(m)},\tfrac{\sqrt{2}}{(m-1)\cdot d(m-1)}\big),$$

we have

$$\lim_{h\to 0} C(h)=193.$$

Remark 3.2. In both Theorems 3.1 and 3.4 the upper bound used for h was $\sqrt{2}/d(1)\approx 5.2769$. However, we observed earlier that $\omega_2^{\varphi}(f;h)=\omega_2^{\varphi}(f;1)$ for $h\ge 1$. If we restrict our attention to values $h\le 1$, then such h is always $<\frac{\sqrt{2}}{3\cdot d(3)}\approx 1.0234574$. That is, we may change "$m\ge 2$" in both theorems mentioned to "$m\ge 4$", and hence the (smaller) relevant bound for the sequence

$$\Gamma(m):=\big(\tfrac{m}{m-1}\big)^2\cdot\tfrac{48}{d^2(m-1)}$$

figuring there will be

$$\sup_{m\ge 4}\big(\tfrac{m}{m-1}\big)^2\cdot\tfrac{48}{d^2(m-1)}=\big(\tfrac{4}{3}\big)^2\cdot\tfrac{48}{d^2(3)}\approx 402.22659.$$

In some instances, it will be enough to consider $0<h\le h_0<1$. In such case it will be possible to increase the lower bound for m to some $m_0=m_0(h_0)$ and thus decrease the upper bound to $\sup_{m\ge m_0}\Gamma(m)$. □

4. THE MAIN RESULT MODIFIED

In this section we present an alternative method to derive estimates for positive linear operators in terms of ω_2^{φ}. The following was communicated to us by Ding-xuan Zhou already early in 1994.

Theorem 4.1. *Let $L:C[a,b]\to C[a,b]$, be a positive linear operator reproducing linear functions, and let $\phi(x)=\sqrt{(x-a)(b-x)}$. Suppose that for some $\alpha>0$ one has*

$$L\big((t-x)^2;x\big)\le d_L\cdot\Big(\frac{\phi(x)}{n^{\alpha}}\Big)^2,$$

where d_L may depend on L. Then

$$\left|L(f;x)-f(x)\right| \le 2\cdot c_4\Big(\lambda, \sqrt{\tfrac{d_L}{2}}\cdot\big(\tfrac{b-a}{2}\big)^{1-\lambda}\Big)\cdot\omega_2^{\phi^\lambda}\Big(f;\sqrt{\tfrac{d_L}{2}}\cdot n^{-\alpha}\cdot\phi(x)^{1-\lambda}\Big) \tag{4.1}$$

for $0 \le \lambda \le 1$. Here $c_4(\lambda, t_0)$ is chosen such that

$$K_2^{\phi^\lambda}(f,t^2) \le c_4(\lambda,t_0)\cdot\omega_2^{\phi^\lambda}(f,t) \quad \textit{for } 0 \le t \le t_0,$$

where $K_2^{\phi^\lambda}$ is defined for $[a,b]$ analogously to (2.2).

Proof. Let $g \in W_{2,\infty}^{\phi^\lambda}[a,b] := \{g : g' \in AC_{\mathrm{loc}}[a,b] \text{ and } \|\phi^{2\lambda}g''\|_\infty < \infty\}$, $x \in (a,b)$. Then

$$\begin{aligned}
|L(g,x)-g(x)| &= \left|L\Big(\int_x^t (t-u)\cdot g''(u)\,\mathrm{d}u, x\Big)\right| \\
&\le L\left(\tfrac{(t-x)^2}{\phi^{2\lambda}(x)};x\right)\cdot\|\phi^{2\lambda}\cdot g''\|_\infty \\
&\le d_L\cdot\tfrac{\phi(x)^{2(1-\lambda)}}{n^{2\alpha}}\cdot\|\phi^{2\lambda}\cdot g''\|_\infty.
\end{aligned}$$

Here we used the fact that $\frac{|t-u|}{\phi^{2\lambda}(u)}$ is monotone for $u \in [x,t]$ or $[t,x]$.

Taking the infimum over $g \in W_{2,\infty}^{\phi^\lambda}[a,b]$, we have for $f \in C[a,b]$, $n \in \mathbb{N}$, $x \in (a,b)$,

$$\begin{aligned}
&|L(f,x)-f(x)| \le \\
&\le \inf_{g\in C^2[a,b]}\Big\{2\cdot\|f-g\|_\infty + d_L\cdot\tfrac{\phi(x)^{2(1-\lambda)}}{n^{2\alpha}}\cdot\|\phi^{2\lambda}\cdot g''\|_\infty\Big\} \\
&= 2\cdot\inf_{g\in C^2[a,b]}\Big\{\|f-g\|_\infty + \tfrac{d_L}{2}\cdot\tfrac{\phi(x)^{2(1-\lambda)}}{n^{2\alpha}}\cdot\|\phi^{2\lambda}\cdot g''\|_\infty\Big\} \\
&\le 2\cdot K_2^{\phi^\lambda}\Big(f;\ \tfrac{d_L}{2}\cdot n^{-2\alpha}\cdot\phi(x)^{2(1-\lambda)}\Big) \\
&\le 2\cdot c_4\Big(\lambda, \max_{x,n}\sqrt{\tfrac{d}{2}}\cdot n^{-\alpha}\cdot\phi(x)^{1-\lambda}\Big)\cdot\omega_2^{\phi^\lambda}\Big(f;\sqrt{\tfrac{d}{2}}\cdot n^{-\alpha}\cdot\phi(x)^{1-\lambda}\Big) \\
&\le 2\cdot c_4\Big(\lambda, \sqrt{\tfrac{d}{2}}\cdot\|\phi^{1-\lambda}\|\Big)\cdot\omega_2^{\phi^\lambda}\Big(f;\sqrt{\tfrac{d}{2}}\cdot n^{-\alpha}\cdot\phi(x)^{1-\lambda}\Big) \\
&= 2\cdot c_4\Big(\lambda, \sqrt{\tfrac{d}{2}}\cdot\big(\tfrac{b-a}{2}\big)^{1-\lambda}\Big)\cdot\omega_2^{\phi^\lambda}\Big(f;\sqrt{\tfrac{d}{2}}\cdot n^{-\alpha}\cdot\phi(x)^{1-\lambda}\Big).
\end{aligned}$$

So inequality (4.1) holds. □

COROLLARY 4.2. *For a positive linear operator $L : C[0,1] \to C[0,1]$ reproducing linear functions and satisfying*

$$L\big((t-x)^2,x\big) \le d_L\cdot\Big(\frac{\varphi(x)}{n^\alpha}\Big)^2$$

we have

$$\|Lf-f\|_\infty \le 2\cdot c_4\Big(1,\sqrt{\tfrac{d_L}{2}}\Big)\cdot\omega_2^{\varphi}\Big(f;\sqrt{\tfrac{d_L}{2}}\cdot\tfrac{1}{n^\alpha}\Big).$$

5. APPLICATION TO CLASSICAL BERNSTEIN OPERATORS

In this section we first give an application of the general Theorem 3.1 to classical Bernstein operators. In order to prove a result that parallels the development in the introduction, we choose $h = \frac{1}{\sqrt{n}}$.

COROLLARY 5.1. *If L in Theorem 3.1 is the Bernstein operator B_n and $h = \frac{1}{\sqrt{n}}$, $n \geq 1$, then we have*

$$\|B_n f - f\|_\infty \leq 405 \cdot \omega_2^\varphi\big(f, \tfrac{1}{\sqrt{n}}\big).$$

Proof. It is well-known that $B_n\big((e_1 - x)^2; x\big) = \frac{x(1-x)}{n} = \frac{\varphi^2(x)}{n}$. We are thus lead to

$$\|B_n f - f\|_\infty \leq \Big[2 + \big(\tfrac{m}{m-1}\big)^2 \cdot \tfrac{48}{d^2(m-1)}\Big] \cdot \omega_2^\varphi\big(f, \tfrac{1}{\sqrt{n}}\big),$$

where $m = m(\frac{1}{\sqrt{n}}) \geq 2$ is such that

$$\tfrac{\sqrt{2}}{m \cdot d(m)} \leq \tfrac{1}{\sqrt{n}} < \tfrac{\sqrt{2}}{(m-1)\cdot d(m-1)}.$$

It has to be kept in mind, though, that $1/\sqrt{n} \leq 1$. That is, we may restrict our attention to values $m - 1 \geq 3$, i.e., $m \geq 4$ (cf. Remark 3.1 (i)). Therefore, we obtain

$$\begin{aligned}\|B_n f - f\|_\infty &\leq \Big[2 + \sup_{m \geq 4} \big(\tfrac{m}{m-1}\big)^2 \cdot \tfrac{48}{d^2(m-1)}\Big] \cdot \omega_2^\varphi\big(f, \tfrac{1}{\sqrt{n}}\big) \\ &\leq 405 \cdot \omega_2^\varphi\big(f, \tfrac{1}{\sqrt{n}}\big), \quad n \geq 1. \qquad \square\end{aligned}$$

REMARK 5.1. The limiting constant in a statement akin to that of Corollary 5.1 is

$$\lim_{m \to \infty} 2 + \big(\tfrac{m}{m-1}\big)^2 \cdot \tfrac{48}{d^2(m-1)} = 2 + 1 \cdot \tfrac{48}{\frac{1}{4}} = 194. \qquad \square$$

If L in Corollary 4.2 is substituted by B_n, then we obtain

$$\|B_n f - f\|_\infty \leq 2 \cdot c_4\big(1, \tfrac{1}{\sqrt{2}}\big) \cdot \omega_2^\varphi\big(f, \tfrac{1}{\sqrt{2n}}\big).$$

We thus have to determine a bound for $c_4\big(1, \frac{1}{\sqrt{2}}\big)$, where $\frac{1}{\sqrt{2}} \approx 0.7071067$. As

$$\tfrac{\sqrt{2}}{5 \cdot d(5)} \approx 0.5828223 < 0.7071067 < 0.740754 \approx \tfrac{\sqrt{2}}{4 \cdot d(4)},$$

we get

$$c_4\big(1, \tfrac{1}{\sqrt{2}}\big) \leq 1 + \sup_{m \geq 5} \Gamma(m) < 330.5,$$

whence we conclude the validity of

COROLLARY 5.2.

$$\|B_n f - f\|_\infty \leq 661 \cdot \omega_2^\varphi\big(f, \tfrac{1}{\sqrt{2n}}\big), \quad n \geq 1.$$

Remark 5.2. (i) The constant $405 = 2+403$ from Corollary 5.1 is not directly comparable to 661 from Corollary 5.2, since $\omega_2^{\varphi}\big(f, \frac{1}{\sqrt{n}}\big)$ figures in the former, and $\omega_2^{\varphi}\big(f, \frac{1}{\sqrt{2n}}\big)$ in the latter.

(ii) The limiting constant in inequalities similar to that in Corollary 5.2 is

$$2 \cdot \big(1 + \lim_{m\to\infty} \Gamma(m)\big) = 2 \cdot (1+192) = 386. \qquad \square$$

6. FURTHER APPLICATIONS

6.1. Piecewise linear interpolation. Let $\Delta_n : a = x_0 < x_1 < ... < x_n = b$ be a partition of the interval $[a, b]$, and let S_{Δ_n} be the positive linear operator associating with each $f \in C[a, b]$ the piecewise linear and continuous function interpolating f at x_i, $i = 0, \dots, n$. As was observed by T. Popoviciu [30], S_{Δ_n} can be represented in the following way:

$$\begin{aligned} S_{\Delta_n}(f;x) &= f(x_0) + (x-x_0)\cdot[x_0,x_1;f] \\ &\quad + \sum_{k=2}^{n} \tfrac{x_k-x_{k-2}}{2}\big(|x-x_{k-1}| + x - x_{k-1}\big)\cdot[x_{k-2},x_{k-1},x_k;f], \end{aligned}$$

where $[a, b; f]$ and $[a, b, c; f]$ denote divided differences of f.

Since S_{Δ_n} reproduces linear functions, determining $S_{\Delta_n}\big((t-x)^2;x\big)$ amounts to representing

$$S_{\Delta_n}(e_2,x) - x^2, \quad \text{for } x \in [a,b], \text{ or for } x \in [x_k, x_{k+1}], \quad 0 \le k \le n-1.$$

But for $x \in [x_k, x_{k+1}]$, $S_{\Delta_n}(e_2, x) - x^2$ is just the remainder of linear Lagrange interpolation at x_k and x_{k+1}, and thus

$$S_{\Delta_n}((t-x)^2;x) = (x-x_k)(x_{k+1}-x), \quad \text{for } x \in [x_k, x_{k+1}].$$

The latter is a piecewise quadratic polynomial, and we have to find out its relationship to $\phi^2(x) = (x-a)(b-x)$, depending on the structure of the partition Δ_n. For simplicity we consider again the case $[a, b] = [0, 1]$. Let

$$t_n(u) := (u-x_k)(x_{k+1}-u), \quad \text{for } u \in [x_k, x_{k+1}], \quad 0 \le k \le n-1.$$

We are thus looking for sufficient conditions on Δ_n under which

$$t_n(u) \le C \cdot \tfrac{u(1-u)}{n^{\alpha}}, \quad 0 \le u \le 1,$$

where $C = C(\Delta_n)$ and $\alpha = \alpha(\Delta_n)$ are suitably chosen.

In Theorem 12 of [14] the following necessary and sufficient conditions on Δ_n are given for the existence of positive linear operators solving the so-called "strong form of Butzer's problem":

There exists a sequence of partitions of the interval $[0, \frac{\pi}{2}]$,

$$\delta_n : 0 = \theta_0 < \theta_1 < \ldots < \theta_n = \tfrac{\pi}{2},$$

such that

(i) $x_k = \sin^2\theta_k, \quad k = 0, \dots, n;$

(ii) $\theta_{k+1} - \theta_k \le \frac{c}{n}$, $\quad k = 0, \dots, n-1$, where c is a constant independent of n and k.

Let us define

$$\theta_k = \tfrac{(2k-1)\pi}{4(n-1)}, \quad k = 1, \dots, n-1, \quad n \ge 2, \quad \theta_0 = 0, \quad \theta_n = \tfrac{\pi}{2}.$$

In this case the x_k are obtained from the zeros of the Chebyshev polynomial $T_{n-1}(x) = \cos\big((n-1)\cdot \arccos x\big)$ after a linear transformation of $[-1,1]$ onto the interval $[0,1]$.

We have

$$\theta_{k+1} - \theta_k = \tfrac{\pi}{2(n-1)} = \tfrac{\pi}{2} \cdot \tfrac{n}{n-1} \cdot \tfrac{1}{n} \le \pi \cdot \tfrac{1}{n}, \quad k = 1, \dots, n-2, \quad n \ge 2,$$

as well as

$$\theta_1 - \theta_0 = \theta_n - \theta_{n-1} = \tfrac{\pi}{4(n-1)} = \tfrac{\pi}{4} \cdot \tfrac{n}{n-1} \cdot \tfrac{1}{n} \le \tfrac{\pi}{2} \cdot \tfrac{1}{n}.$$

So we get $c = \pi$ in (ii) above.

Here we point out that in Theorem 9 in [22] sufficient conditions on the nodes are given, such that a certain operator satisfies the DeVore-Gopengauz inequality.

The nodes

$$y_k = 2x_k - 1, \quad k = 0, \dots, n,$$

satisfy these conditions if the range $[-1,1]$ is considered instead of $[0,1]$. For our choice of θ_k it is easy to compute that in Theorem 9 in [22] we can take $c = 2\pi$ and $\beta = 3$.

We go on with the estimate of the function

$$g_n(u) := \tfrac{t_n(u)}{u(1-u)}, \quad u \in [x_k, x_{k+1}], \quad 0 \le k \le n-1.$$

First we consider $1 \le k \le n-2$. It is easy to compute that

$$\max_{u\in[x_k,x_{k+1}]} g_n(u) = \sin^2(\theta_{k+1} - \theta_k) \le (\theta_{k+1} - \theta_k)^2 = \left(\tfrac{\pi}{2(n-1)}\right)^2 \le \pi^2 \cdot \tfrac{1}{n^2}.$$

Let $u \in [0,x]$. Then

$$g_n(u) = \tfrac{u(x_1-u)}{u(1-u)} \le x_1 \le \theta_1^2 \le \left(\tfrac{\pi}{2}\right)^2 \cdot \tfrac{1}{n^2}.$$

The same holds for $u \in [x_{n-1}, 1]$.

Similar estimates for $g_n(u)$ are given in [22], but without explicit description of the constants appearing in the proof.

We thus proved that

$$S_{\Delta_n}\big((t-x)^2; x\big) \le \pi^2 \cdot \tfrac{u(1-u)}{n^2}.$$

Setting $h = \frac{1}{n}$, $n \ge 2$, and choosing $m_0 = m_0(n) \in \mathbb{N}$ such that

$$h \in \left[\tfrac{\sqrt{2}}{m_0\cdot d(m_0)}, \tfrac{\sqrt{2}}{(m_0-1)\cdot d(m_0-1)}\right],$$

as a straightforward corollary from Theorem 3.1 we obtain the following

Corollary 6.1. *For $n \geq 2$ and $S_{\Delta_n}(f;x)$, the piecewise linear function interpolating f at the nodes $\{x_k\}$ defined above, we have*

$$\|S_{\Delta_n} f - f\|_\infty \leq c_n \cdot \omega_2^\varphi\big(f, \tfrac{1}{n}\big),$$

where

$$c_n = 2 + \big(\tfrac{m_0}{m_0-1}\big)^2 \cdot \tfrac{48\pi^2}{d^2(m_0-1)} \longrightarrow 2 + 192 \cdot \pi^2, \quad n \to \infty.$$

Next we consider a different way to obtain an estimate for $\big|S_{\Delta_n}(f;x) - f(x)\big|$, i.e., one with a constant smaller than c_n, and without using Theorem 3.1. We follow the proof of (3.8)–(3.10). For $k = 1, \ldots, n-2$, $x \in [x_k, x_{k+1}]$, in the same way as there we verify that

$$\begin{aligned}\big|S_{\Delta_n}(f;x) - f(x)\big| \leq \sup\Big\{&\big|f(x+h) - 2f(x) + f(x-h)\big|,\\ &x, x \pm h \in [x_k, x_{k+1}],\ |h| \leq \tfrac{x_{k+1}-x_k}{2} = \tfrac{y_{k+1}-y_k}{4}\Big\},\end{aligned}$$

where y_k are the zeros of $T_{n-1}(x)$.

To estimate $y_{k+1} - y_k$ we use (7.8) in Chapter 8 of [5]. We get

$$|h| \leq \tfrac{9\pi}{4} \cdot \tfrac{\sqrt{1-y^2}}{n} = \tfrac{9\pi}{2} \cdot \tfrac{\sqrt{x(1-x)}}{n},$$

where $y = 2x - 1$. Hence

$$\|S_{\Delta_n} f - f\|_\infty \leq \omega_2^\varphi\big(f; \tfrac{9\pi}{2n}\big), \quad \text{for } x \in [x_1, x_{n-1}].$$

An analogous estimate holds for $x \in [0, x_1] \cup [x_{n-1}, 1]$. To the best of our knowledge it is not known how to take the constant $\frac{9\pi}{2}$ out of the modulus ω_2^φ without using Theorem 3.1, while the latter leads to an enormous increase of the constant multiplying ω_2^φ.

Considering the last inequality and Corollary 6.1 the following question arises: Can we obtain an estimate of the type

$$\|S_{\Delta_n} f - f\|_\infty \leq \beta \cdot \omega_2^\varphi\big(f; \tfrac{\gamma}{n}\big)$$

with positive constants β and γ as small as possible?

In the next to the last inequality we have $\gamma = 9\pi/2$, $\beta = 1$, and in Corollary 6.1 we got $\gamma = 1$, $\beta = c_n \to 2 + 192 \cdot \pi^2$. It is clear that a smaller value of β leads to a bigger one of γ and vice versa.

Remark 6.1. The number of nodes $\{x_k\}$ of $S_{\Delta_n}(f;x)$ is $n+1$ in this section, while the number of nodes in Lemma 3.2 is $\mathcal{O}(n^2)$, if we write $m = n$ there. Obviously, increasing the number of nodes, i.e., using $\{y_k\}$ from Lemma 3.2 instead of $\{x_k\}$, leads to a constant better than $9\pi/2$, obtained in the second estimate of this section. $\square$

6.2. Bernstein-Stancu operators. In the article [32] published in 1972, D. D. Stancu introduced a multiparameter generalization of the classical Bernstein operator which was further investigated, generalized and modified in some 40 papers since then. One recent contribution is due to Stancu himself (see [35]), who presented certain even more general mappings $L_{n,p,r}^{\alpha,\beta,\gamma}$, thus unifying several earlier approaches.

In this section we focus on those cases of the above operators which preserve e_0 and e_1, namely $L_{n,0,r}^{\alpha,0,0}$, and which we will write as $L_{n,r}^{\alpha}$ for brevity. These were first investigated in [33] and [34], and are given as follows.

Let r be a non-negative integer parameter, n is a natural number such that $n > 2r$, while α is a non-negative parameter which may depend on n. To each $f : [0,1] \to \mathbb{R}$ we associate

$$L_{n,r}^{\alpha}(f;x) := \sum_{k=0}^{n-r} p_{n-r,k}^{\langle\alpha\rangle}(x) \cdot \left\{ \left[1 - x + (n-r-k)\alpha\right] \cdot f\left(\tfrac{k}{n}\right) + (x + k\alpha) \cdot f\left(\tfrac{k+r}{n}\right) \right\},$$

where, in terms of factorial powers

$$t^{[m,h]} := t(t-h)\dots(t-(m-1)h), \quad t^{[0,h]} := 1,$$

we have

$$p_{n-r,k}^{\langle\alpha\rangle}(x) := \binom{n-r}{k} \cdot \frac{x^{[k,-\alpha]}(1-x)^{[n-r-k,-\alpha]}}{(1+\alpha)^{[n-r,-\alpha]}}.$$

For $(\alpha, r) = (0,0)$ and $(\alpha, r) = (0,1)$ the operator becomes the classical Bernstein operator. For $\alpha \geq 0$ and r given as above, it is a positive linear operator.

Stancu showed that

$$\begin{aligned} L_{n,r}^{\alpha} e_i &= e_i, \quad i = 0,1, \quad \text{and that} \\ L_{n,r}^{\alpha}(e_2;x) - x^2 &= L_{n,r}^{\alpha}((e_1 - x)^2; x) \\ &= \left[1 + \alpha n + \tfrac{r(r-1)}{n}\right] \cdot \tfrac{x(1-x)}{n(1+\alpha)} =: d_{L_{n,r}^{\alpha}} \cdot \left(\tfrac{\varphi(x)}{n^{1/2}}\right)^2. \end{aligned}$$

We apply Theorem 3.1 first, putting there again $h = \frac{1}{\sqrt{n}}$, $n \geq 1$. This gives

$$\left|L_{n,r}^{\alpha}(f;x) - f(x)\right| \leq \left[2 + \left(\tfrac{m}{m-1}\right)^2 \cdot \tfrac{48}{d^2(m-1)} \cdot d_{L_{n,r}^{\alpha}}\right] \cdot \omega_2^{\varphi}\left(f; \tfrac{1}{\sqrt{n}}\right).$$

As in the proof of Corollary 5.1, we note that we can restrict our attention to values $m \geq 4$. This implies, using $\Gamma(m)$ from Remark 3.2 again,

$$\begin{aligned} \left\|L_{n,r}^{\alpha} f - f\right\|_{\infty} &\leq \left[2 + \sup_{m \geq 4} \Gamma(m) \cdot \tfrac{1}{1+\alpha} \cdot \left(1 + \alpha n + \tfrac{r(r-1)}{n}\right)\right] \cdot \omega_2^{\varphi}\left(f; \tfrac{1}{\sqrt{n}}\right) \\ &\leq \left[2 + \tfrac{403}{1+\alpha} \cdot \left(1 + \alpha n + \tfrac{r(r-1)}{n}\right)\right] \cdot \omega_2^{\varphi}\left(f; \tfrac{1}{\sqrt{n}}\right). \end{aligned}$$

Furthermore, if $0 \leq \alpha = \alpha(n) \leq A \cdot \frac{1}{n}$, then

$$\left\|L_{n,r}^{\alpha} f - f\right\|_{\infty} \leq \left[2 + 403 \cdot \left(1 + A + \tfrac{r(r-1)}{n}\right)\right] \cdot \omega_2^{\varphi}\left(f; \tfrac{1}{\sqrt{n}}\right), \quad n \geq 1.$$

For $(\alpha, r) = (0,0)$ or $(\alpha, r) = (0,1)$ we arrive again at the statement of Corollary 5.1.

Applying Corollary 4.2 yields

$$\left\|L_{n,r}^{\alpha} f - f\right\|_{\infty} \leq 2 \cdot c_4\Big(1, \sqrt{\tfrac{1}{2} d_{L_{n,r}^{\alpha}}}\Big) \cdot \omega_2^{\varphi}\Big(f; \sqrt{\tfrac{1}{2n} d_{L_{n,r}^{\alpha}}}\Big).$$

Here

$$\tfrac{1}{2} d_{L_{n,r}^{\alpha}} = \tfrac{1}{2(1+\alpha)} \cdot \big(1 + \alpha n + \tfrac{r(r-1)}{n}\big).$$

It is meaningful to also assume here that $\alpha = \alpha(n) \leq A \cdot \frac{1}{n}$, so that

$$\tfrac{d_{L_{n,r}^{\alpha}}}{2} \leq \tfrac{1}{2}[1 + A + r(r-1)],$$

whence

$$\left\|L_{n,r}^{\alpha} f - f\right\|_{\infty} \leq 2 \cdot c_4\Big(1, \sqrt{\tfrac{1+A+r(r-1)}{2}}\Big) \cdot \omega_2^{\varphi}\left(f; \sqrt{1 + A + r(r-1)} \cdot \tfrac{1}{\sqrt{2n}}\right).$$

For $(\alpha, r) = (0,0)$ or $(\alpha, r) = (0,1)$ we have again

$$\begin{aligned} \left\|B_n f - f\right\|_{\infty} &\leq 2 \cdot c_4\big(1, \tfrac{1}{\sqrt{2}}\big) \cdot \omega_2^{\varphi}\big(f; \tfrac{1}{\sqrt{2n}}\big) \\ &\leq 661 \cdot \omega_2^{\varphi}\big(f; \tfrac{1}{\sqrt{2n}}\big) \quad \text{(cf. Corollary 5.2)} \end{aligned} \qquad \square$$

Inequalities similar to the ones from this section can also be obtained for a certain class of Bernstein-type operators introduced and investigated by Brass [1]. These are related to the above Bernstein–Stancu operators and also generalize other operators. For further results on Brass operators see [25], [26] and [27].

6.3. Gavrea operators. In 1996 Gavrea published the article [13] in which a long-standing problem on positive linear operators was solved. Among other things, he introduced a sequence of positive linear polynomial operators $H_{2n+1} : C[0,1] \to \Pi_{2n+1}$ which reproduce linear functions and for which one has

$$\begin{aligned} H_{2n+1}((e_1 - x)^2; x) &\leq x(1-x) \cdot (1 - x_n) \\ &\leq x(1-x) \cdot \tfrac{K}{n^2}. \end{aligned}$$

Here $x_n < 1$ is the largest root of the Jacobi polynomial $J_n^{(1,0)}$ (defined on $[0,1]$) and K is a constant independent of n. A numerical value for K was not given in Gavrea's paper. We refrain from giving the rather complicated definition of H_{2n+1} here and refer the reader instead to [13].

Applying Theorem 3.1, using $h = \frac{1}{n}$ now, shows that

$$\|H_{2n+1} f - f\|_{\infty} \leq \big(2 + \Gamma(m) \cdot K\big) \cdot \omega_2^{\varphi}\big(f, \tfrac{1}{n}\big),$$

where $\frac{\sqrt{2}}{m \cdot d(m)} \leq \frac{1}{n} < \frac{\sqrt{2}}{(m-1) \cdot d(m-1)}$.

Again it suffices to consider $m \geq 4$ so that we arrive at

$$\|H_{2n+1} f - f\|_{\infty} \leq (2 + 403 \cdot K) \cdot \omega_2^{\varphi}\big(f, \tfrac{1}{n}\big), \quad n \geq 1.$$

It is also possible to apply Theorem 4.1 to H_{2n+1} in order to derive similar estimates in terms of $\omega_2^{\varphi^\lambda}$, $0 \le \lambda \le 1$.

Acknowledgment. A big "thank you" from Sofia and Duisburg to Jutta Gonska at Mainz for her superb preparation of the final manuscript. We also gratefully acknowledge the help of Laura Beutel (Braşov and Duisburg).

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Received by the editors: September 26, 2001.