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ON THE CAUCHY TRANSFORM AND COMPLEX CUBIC SPLINES

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Abstract. In this paper one approximates the Cauchy transform of a complex function on a simple closed curve, using an interpolation cubic spline function given by Iancu (1987).

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1. INTRODUCTION

Atkinson (1972) considered the Cauchy transform

$$Tf(z) = \frac{1}{\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta, \quad z \in \Gamma, \tag{1}$$

where Γ is a simple closed curve, in the complex plane, with continuously differentiable parametrization. For the evaluation of $Tf(z)$ there are investigated numerical methods which are based on replacing $f(z)$ by a uniformly convergent sequence $\{\Phi_n(z) \mid n \geq 1\}$ to $f(z)$. Based on the fact that, if Φ_n converge to f, then $T\Phi_n$ converge to Tf with the same speed, Atkinson (1972) has studied the cases in which the functions $\Phi_n(z)$ are defined as piecewise linear and piecewise quadratic interpolation function to $f(z)$ at a given set of node points on Γ.

In this paper we give an extension of Atkinson's results for the case when $\Phi_n(z)$ are taken as interpolating complex cubic spline functions for f on the Γ curve, and, also, an extension of our previous results (Iancu, 1987; 1989).

2. THE INTERPOLATION CUBIC SPLINE FUNCTION

Let Γ be a closed rectifiable curve in the complex plane. On Γ curve one considers the partition

$$\Delta_\Gamma : \{P_0, P_1, \ldots, P_n, P_{n+1} \mid P_0 \equiv P_{n+1}\}, \tag{2}$$

of which points are arranged in trigonometric direction. The points of partition (2) divides the curve Γ in the arcs Γ_k from P_{k-1} to P_k, $k = 1, 2, \ldots, n+1$.

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One denotes

$$h_k = z_k - z_{k-1}, \quad k = 1, 2, \ldots, n+1, \tag{3}$$

where $z_k = x_k + \mathrm{i}y_k$, $x_k, y_k \in \mathbb{R}$, is the affix of P_k, $k = 1, 2, \ldots, n+1$ (obviously $z_0 = z_{n+1}$). Let f be a continuous function on Γ, about which we know that it takes the values:

$$f_k = f(z_k), \quad k = 0, 1, 2, \ldots, n+1. \tag{4}$$

Let us consider the interpolating function of the form (Iancu, 1987):

$$\begin{aligned} \Phi_n(z) &= \tfrac{M_k - M_{k-1}}{6h_k}(z - z_{k-1})^3 + \tfrac{M_{k-1}}{2}(z - z_{k-1})^2 + \\ &\quad + m_{k-1}(z - z_{k-1}) + f_{k-1}, \end{aligned} \tag{5}$$

with $z \in \Gamma_k$ and $z \in [z_{k-1}, z_k]$, $k = 1, 2, \ldots, n+1$. The function $\Phi_n(z)$ is uniquely determined by the conditions system:

$$\begin{aligned} M_k &= 6\tfrac{f_k - f_{k-1}}{h_k^2} - 6\tfrac{m_{k-1}}{h_k} - 2M_{k-1}, \\ m_k &= 3\tfrac{f_k - f_{k-1}}{h_k} - 2m_{k-1} - \tfrac{M_{k-1}}{2}h_k, \quad k = \overline{1, n+1} \\ M_0 &= M_{n+1}, \quad m_0 = m_{n+1}, \end{aligned} \tag{6}$$

with $M_j = \Phi_n''(z_j)$, $m_j = \Phi_n'(z_j)$, $j = 1, 2, \ldots, n+1$.

3. THE ERROR ANALYSIS

Next, we prove the following:

LEMMA 1. *Let be the complex spline function* (5) *determined by the conditions* (6). *For any function $f \in C(\Gamma)$ that is interpolated by the spline function* (5), *we have:*

$$\|f - \Phi_n\| \leq 6\,\omega(f; h) + 2h(2|m_{k-1}| + |m_k|), \tag{7}$$

where

$$\omega(f, h) = \sup_{\substack{|w_1 - w_2| \leq h \\ w_1,\, w_2 \in \Gamma}} |f(w_1) - f(w_2)| \tag{8}$$

is the modulus of continuity of f on Γ and $h = \max\limits_{k \in \{1,2,\ldots,n+1\}} |h_k|$.

Proof. On the base of uniform norm definition, we have:

$$\|f - \Phi_n\|_\infty = \sup_{z \in \Gamma} |f(z) - \Phi_n(z)|. \tag{9}$$

By using (5), one obtains:

$$\begin{aligned} f(z) - \Phi_n(z) &= f(z) - \tfrac{M_k - M_{k-1}}{6h_k}(z - z_{k-1})^3 - \tfrac{M_{k-1}}{2}(z - z_{k-1})^2 - \\ &\quad - m_{k-1}(z - z_{k-1}) - f_{k-1}, \end{aligned}$$

with $z \in \Gamma_k$ and $z \in [z_{k-1}, z_k]$, $k = 1, 2, \ldots, n+1$. With aid of system (6), having in view the definition of the modulus of continuity (8), the expression of the interpolating cubic spline (5), by using Eq. (9), we can give a simple formula for the difference $f(z) - \Phi_n(z)$. So, we have

$$\begin{aligned} f(z) - \Phi_n(z) \;&=\; f(z) - \tfrac{(z-z_{k-1})^3}{6h_k}\left(-12\tfrac{f_k - f_{k-1}}{h_k^2} + 6\tfrac{m_{k-1}}{h_k} + 6\tfrac{m_k}{h_k}\right) - \\ &- \tfrac{(z-z_{k-1})^2}{2}\left(6\tfrac{f_k - f_{k-1}}{h_k^2} - 4\tfrac{m_{k-1}}{h_k} - 2\tfrac{m_k}{h_k}\right) - \\ &- m_{k-1}(z - z_{k-1}) - f_{k-1}, \quad k = 1, 2, \ldots, n+1. \end{aligned}$$

From here, one obtains:

$$\begin{aligned} (10) \qquad |f(z) - \Phi_n(z)| \;&\leq\; \omega(f;h) + 2\tfrac{\omega(f;h)h^3}{|h_k|^3} + \tfrac{|m_{k-1}|h^3}{|h_k|^2} + \tfrac{|m_k|h^2}{|h_k|^3} + \\ &+ 3\tfrac{\omega(f;h)h^2}{|h_k|^2} + 2\tfrac{|m_{k-1}|h^2}{|h_k|} + \tfrac{|m_k|h^2}{|h_k|} + |m_{k-1}|h. \end{aligned}$$

If the node points of the partition (2) are taken so that $|h_k| = h$, then, from (10), results:

$$\begin{aligned} (11) \qquad |f(z) - \Phi_n(z)| &\leq \omega(f;h) + 2\omega(f;h) + |m_{k-1}|h + |m_k|h + \\ &+ 3\omega(f;h) + 2|m_{k-1}|h + |m_k|h + |m_{k-1}|h \\ &= 6\,\omega(f;h) + 2\,h(2\,|m_{k-1}| + |m_k|). \end{aligned}$$

□

REMARK 1. Because the system (6) determines uniquely the complex values M_j and m_j, where $j = 1, 2, \ldots, n$, the formula (9) can be written in the form:

$$||f - \Phi_n|| \leq \omega(f;h) + h\left(\tfrac{h}{6}|M_k| + \tfrac{2h}{3}|M_{k-1}| + |m_{k-1}|\right).$$

□

Let the Banach space $H^\mu(\Gamma)$ of the functions which satisfy the Hölder condition

$$|f(t'') - f(t')| \leq A|t - t'|^\mu, \quad \forall\, t', t'' \in \Gamma, \quad \mu \in (0,1],$$

where A is the Hölder constant and $\mu \in (0,1]$ is the Hölder exponent.

We have the following result:

THEOREM 1. *Let be the functions f and $\{\Phi_n,\ n \geq 1\}$ from Lemma 1. If $f, \Phi_n \in H^\mu(\Gamma)$, $\mu \in (0,1]$ and if, for all t', $t'' \in \Gamma_k$, $k = 1, 2, \ldots, n+1$ is fulfilled the condition $|t' - t''| < h$, then*

$$||f - \Phi_n||_\mu \leq A + 6\omega(f;h) + 4h|m_{k-1}| + |m_k|(2h + h^{1-\mu}).$$

Proof. Taking into account the results of Atkinson (1972), Chien-Ke Lu (1982), Muskhelishvilli (1953) we obtain:

$$||f - \Phi_n||_\mu = ||f - \Phi_n||_\infty + M_\mu(f - \Phi_n),$$

where

$$M_\mu(f - \Phi_n) = \sup_{\substack{t', t'' \in \mathbb{R} \\ t' \neq t''}} \frac{\left|\big(f(t') - \Phi_n(t')\big) - \big(f(t'') - \Phi_n(t'')\big)\right|}{|t' - t''|^\mu}.$$

By using (5), and taking into account the conditions of the Theorem 1 we have obtained the evaluation of $M_\mu(f-\Phi_n)$. So we have:

$$
\begin{aligned}
(12)\ E &= \Big|\big(f(t')-\Phi_n(t')\big)-\big(f(t'')-\Phi_n(t'')\big)\Big| \\
&\le \big|f(t')-f(t'')\big|+\big|\Phi_n(t'')-\Phi_n(t')\big| \\
&\le A|t'-t''|^\mu+\Big|\tfrac{M_k-M_{k-1}}{6h_k}(t''-t')\big[(t''-z_{k-1})^2+ \\
&\quad +(t''-z_{k-1})(t'-z_{k-1})+(t'-z_{k-1})^2\big]+ \\
&\quad +\tfrac{1}{2}M_{k-1}(t''-t')\big[(t''-z_{k-1})+(t'-z_{k-1})\big]+m_{k-1}(t''-t')\Big|.
\end{aligned}
$$

Considering the case when the node points are equidistant $|h_k|=h$, $k=1,2,\dots,n+1$ and taking into account (6), we have:

$$
\begin{aligned}
(13)\quad & \\
E &\le A|t'-t''|^\mu+|t''-t'|\left|\tfrac{M_k-M_{k-1}}{6h}\,(h^2+hh+h^2)+\tfrac{1}{2}M_{k-1}(h+h)+m_{k-1}\right| \\
&= A|t'-t''|^\mu+|t''-t'|\,|m_k|.
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
M_\mu(f-\Phi_n) &= \sup_{\substack{t',t''\in\Gamma_k\\ t'\neq t''}} \frac{E}{|t'-t''|^\mu} \\
(14)\qquad &\le \sup_{\substack{t',t''\in\Gamma_k\\ t'\neq t''}} \frac{A|t'-t''|^\mu+|t''-t'||m_k|}{|t'-t''|^\mu} \\
&= A+|m_k|h^{1-\mu}
\end{aligned}
$$

and, so, the theorem is proved. □

4. THE CAUCHY TRANSFORM OF Φ_n

We have the following proposition:

PROPOSITION 1. *Let us consider a complex cubic spline function of the form* (5) *in the conditions* (6). *Then, for the Cauchy transform* $T\Phi_n(z)$, *by using the formula* (1), *we obtain the following result:*

$$
T\Phi_n(z)=\frac{1}{\pi \mathrm{i}}\sum_{k=1}^{n+1}[T_{k,1}(z)\;+\;T_{k,2}(z)\;+\;T_{k,3}(z)], \tag{15}
$$

where

$$
\begin{aligned}
T_{k,1}(z) &= \tfrac{M_k - M_{k-1}}{6h_k}\Big\{h_k\big[\tfrac{1}{3}(z_k - z)^2 + \tfrac{7}{6}(z - z_{k-1})(z_{k-1} + z_k - 2z) + \\
&\quad + 3(z - z_{k-1})^2\big] + (z - z_{k-1})^3\, I_k(z)\Big\},\\
T_{k,2}(z) &= \tfrac{M_{k-1}}{2}\,\Big\{h_k\big[\tfrac{z_{k-1}+z_k-2z}{2} + 2(z - z_{k-1})\big] + (z - z_{k-1})^2\, I_k(z)\Big\}\,,\\
T_{k,3}(z) &= m_{k-1}\big[h_k + (z - z_{k-1})\, I_k(z)\big] + f_{k-1}\, I_k(z),\\
I_k(z) &= \int_{z_{k-1}}^{z_k} \tfrac{\mathrm{d}\zeta}{\zeta - z}.
\end{aligned}
$$

5. NUMERICAL RESULTS

For the third function in the example set 1 (Atkinson, 1972), we obtained the results given in Table 1 (see Table 2 of Atkinson (1972), p. 295). In that example, the contour Γ is an ellipse with the center at the origin, given by $z(t) = \cos t + \mathrm{i}\lambda \sin t$, where $t \in (0, 2\pi)$ and $\lambda \in (0, 1]$.

	Error, $\lambda = 1.0$		Error, $\lambda = 0.2$	
n	Atkinson (1972)	Present paper	Atkinson (1972)	Present paper
32	$9.59 \cdot 10^{-3}$	$1.59 \cdot 10^{-4}$	$6.44 \cdot 10^{-2}$	$3.67 \cdot 10^{-4}$
64	$1.18 \cdot 10^{-3}$	$1.36 \cdot 10^{-5}$	$1.49 \cdot 10^{-2}$	$3.61 \cdot 10^{-6}$
128	$1.59 \cdot 10^{-4}$	$1.02 \cdot 10^{-6}$	$2.61 \cdot 10^{-3}$	$3.30 \cdot 10^{-9}$

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