REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 32 (2003) no. 1, pp. 69–71
ictp.acad.ro/jnaat

ON THE OPTIMAL SELECTION OF THE RELAXATION CONSTANT IN THE JOR METHOD

SLOBODAN LAKIĆ* and FERENC SZIDAROVSZKY†

Abstract. In this paper we suggest a method for the quasi-optimal selection of the relaxation constant in the Jacobi overrelaxation (JOR) method. It is assumed that the eigenvalues of the coefficient matrix belong to a rectangular region. Our estimates may lead to better parameter choices than earlier results on circular regions.

MSC 2000. 65F10.
Keywords. Overrelaxation, eigenvalue bounds, speed of convergence.

1. INTRODUCTION

Consider a system of linear equations $Ax = b$, where $A \in \mathbb{C}^{m,m}$ and $x, b \in \mathbb{C}^m$. It is assumed that A is nonsingular and has nonzero diagonal entries. Without losing generality we can suppose that $a_{ii} = 1$, $i = 1, 2, \ldots, m$. The associated JOR method has the form as in [2]

$$x_{n+1} = Mx_n + d, \quad x_0 \in \mathbb{C}^m$$

with $M = I - rA$, $d = rb$, where r is a nonzero real parameter. In an earlier paper [1], Lakić has introduced a new method for the quasioptimal selection of parameter r by assuming that all eigenvalues of the coefficient matrix A belong to a certain circular region. In this paper we will present two analogous results for rectangular regions and we will also demonstrate with a numerical example that rectangular regions may lead to sharper estimates.

2. QUASI-OPTIMAL SELECTION OF r

Assume that all eigenvalues of matrix A belong to a rectangular region

$$S = \{\lambda \mid \alpha \leq \operatorname{Re}\lambda \leq \beta,\ -\gamma \leq \operatorname{Im}\lambda \leq \gamma\}$$

where $\gamma \geq 0$, and α, β are real. If $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of A then the eigenvalues of $M = I - rA$ are the numbers $z_k = 1 - r\lambda_k$, $k = 1, 2, \ldots, m$. Notice that $|z_k|^2 = (1 - ra_k)^2 + (rb_k)^2$, where $\lambda_k = a_k + ib_k$. Two special cases will be next discussed and the quasi-optimal selection of r will be presented.

*University of Novi Sad, Technical Faculty "Mihajlo Pupin", 23000 Zrenjanin, Yugoslavia.
†Systems and Industrial Engineering Department, University of Arizona, Tucson, Arizona, 85721-0020, U.S.A., e-mail: szidar@sie.arizona.edu.

THEOREM 1. *Assume that*

$$0 < \alpha \le \beta \le \alpha + \tfrac{2\gamma^2}{\alpha}.$$

Then with the selection of

$$r = \frac{\alpha}{\alpha^2 + \gamma^2}$$

we obtain

$$\rho(M) \le \tfrac{\gamma}{\sqrt{\alpha^2+\gamma^2}} < 1.$$

Proof. The conditions imply that $r(\alpha + \beta) \le 2$, therefore $r(a_k + \alpha) \le 2$. This relation is equivalent to the inequality

$$(1 - ra_k)^2 \le (1 - r\alpha)^2.$$

Notice that

$$|z_k| \le \sqrt{(1 - r\alpha)^2 + (r\gamma)^2} = f(r)$$

with

$$f'\big(\tfrac{\alpha}{\alpha^2+\gamma^2}\big) = 0, \qquad f''\big(\tfrac{\alpha}{\alpha^2+\gamma^2}\big) > 0.$$

Therefore $f(r)$ is minimal at $r = \frac{\alpha}{\alpha^2+\gamma^2}$ with value

$$f\big(\tfrac{\alpha}{\alpha^2+\gamma^2}\big) = \tfrac{\gamma}{\sqrt{\alpha^2+\gamma^2}} < 1.$$

□

THEOREM 2. *Assume that* $0 < \alpha \le \beta$. *Then with the selection of*

$$r = \frac{\alpha}{\beta^2 + \gamma^2}$$

we obtain

$$\rho(M) \le \sqrt{1 - \tfrac{\alpha^2}{\beta^2+\gamma^2}} < 1.$$

Proof. Since $\alpha \le a_k \le \beta$ and $|b_k| \le \gamma$, we have

$$|z_k| \le \sqrt{(1 - ra_k)^2 + (r\gamma)^2} \le \sqrt{1 - 2r\alpha + (\beta^2 + \gamma^2)r^2} = g(r).$$

Function g has its minimum at $r = \frac{\alpha}{\beta^2+\gamma^2}$ and with this selection

$$\rho(M) \le \sqrt{1 - \tfrac{\alpha^2}{\beta^2+\gamma^2}} < 1.$$

□

REMARK 1. Inequality $\rho(M) < 1$ implies that the JOR method is globally convergent. □

REMARK 2. Since for all $0 < \alpha \le \beta$,

$$\sqrt{1 - \tfrac{\alpha^2}{\beta^2+\gamma^2}} \ge \tfrac{\gamma}{\alpha^2+\gamma^2},$$

we reach the following conclusions: If

$$0 < \alpha \le \beta \le \alpha + \tfrac{2\gamma^2}{\alpha},$$

then we can use Theorem 1, and if

$$\beta > \alpha + \frac{2\gamma^2}{\alpha}$$

then Theorem 2 can be applied. □

3. NUMERICAL EXAMPLE

Consider the following complex matrix

$$A = \begin{bmatrix} 1 & i & 0 \\ 1 & 1 & i \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{1-i}{2}$, $\lambda_3 = \frac{3+i}{2}$. The best choice for the JOR method is $r = 1$, which is the classical Jacobi method and $\rho(M) = 0.707$. From Theorem 1 (with the selection of $\alpha = \gamma = \frac{1}{2}, \beta = \frac{3}{2}$) we obtain the optimal choice $r = 1$ again with $\rho(M) \leq 0.707$. From Theorem 2 we obtain $r = 0.2$ and the estimation $\rho(M) \leq 0.95$. The true value is $\rho(M) = 0.905$.

In applying the results of Lakić in [1] for the same matrix we select $t = 1 - \frac{1}{\sqrt{2}} = 0.293$ and $T = 1 + \frac{1}{\sqrt{2}} = 1.707$.

Then Theorems 1 and 2 of that earlier paper imply that $r = 0.5$ and $r = 0.1$ are the optimal selection with corresponding estimates $\rho(M) \leq 0.92$ (the exact value is 0.79) and $\rho(M) \leq 0.98$ (the exact value is 0.95), respectively. This results show that the use of rectangular regions might lead to better estimates than the use of circular regions.

REFERENCES

[1] Lakić, S., *A Quasi-Optimum JOR Method*, Rev. Anal. Numér. Théor. Approx., **24** nos. 1–2, pp. 165–167, 1995.

[2] Young, D. M., *Iterative Solution of Large Linear Systems*, Academic Press, New York/London, 1971.

Received by the editors: December 11, 1996.