Abstract. We provide sufficient conditions for the convergence of the Steffensen method for solving the scalar equation \( f(x) = 0 \), without assuming differentiability of \( f \) at other points than the solution \( x^* \). We analyze the cases when the Steffensen method generates two sequences which approximate bilaterally the solution.

MSC 2000. 65H05.

Keywords. Aitken–Steffensen iterations.

1. INTRODUCTION

In this paper we consider the Steffensen method for approximating the solutions of the equations of the form

\[
    f(x) = 0
\]

with \( f : [a, b] \to \mathbb{R} \), \( a, b \in \mathbb{R} \), \( a < b \). Let \( g : [a, b] \to \mathbb{R} \) be such that the equation

\[
    x - g(x) = 0
\]

is equivalent to (1).

As it is well known, the Steffensen method consists in approximating the solution \( x^* \) of (1) by the sequence \( (x_n)_{n \geq 1} \) given by

\[
    x_{n+1} = x_n - \frac{f(x_n)}{g(x_n) - f(x_n)} , \quad n = 1, 2, \ldots , \quad x_0 \in [a, b].
\]

We are interested in the following in the conditions under which the sequences \( (x_n)_{n \geq 1} \) and \( (g(x_n))_{n \geq 1} \) are monotone, and offer bilateral approximations to \( x^* \). The importance of such sequences resides in the fact that at each iteration step we obtain a rigorous error bound. We shall construct the function \( g \) without assuming that \( f \) is differentiable on the whole interval \([a, b]\). In this sense, we shall use the divided differences of \( f \).

Regarding the monotony and convexity of the function \( f \) we shall adopt the following definitions.

\*This work has been supported by the Romanian Academy under grant GAR 19/2003.
\ †“T. Popoviciu” Institute of Numerical Analysis, P. O. Box 68-1, 3400 Cluj-Napoca, Romania, e-mail: pavaloiu@ictp.acad.ro.
**Definition 1.** The function $f$ is nondecreasing (increasing) on $[a, b]$ if
$[u, v; f] \geq 0 \ (> 0) \ \forall u, v \in [a, b]$, while $f$ is nonincreasing (decreasing) if
$[u, v; f] \leq 0 \ (< 0) \ \forall u, v \in [a, b]$.

**Definition 2.** The function $f$ is nonconcave (convex) on $[a, b]$ if
$[u, v, w; f] \geq 0 \ (> 0) \ \forall u, v, w \in [a, b]$, and is nonconvex (concave) if
$[u, v, w; f] \leq 0 \ (< 0) \ \forall u, v, w \in [a, b]$.

Consider the function $p_{x_0} : [a, b] \ {x_0} \rightarrow \mathbb{R}$ given by

$$p_{x_0} = [x_0, x; f].$$

Recall the following result:

**Theorem 3.** [3, p. 290].

a) If $f$ is nonconcave on $[a, b]$ then $p_{x_0}$ is nondecreasing on $[a, b]$;
b) If $f$ is convex on $[a, b]$ then $p_{x_0}$ is increasing on $[a, b]$;
c) If $f$ is nonconvex on $[a, b]$ then $p_{x_0}$ is nonincreasing on $[a, b]$;
d) If $f$ is concave on $[a, b]$ then $p_{x_0}$ is decreasing on $[a, b]$.

Consider now $u, v, w, t \in [a, b]$ such that $u \leq \min\{v, w, t\}$ and $t \geq \max\{u, v, w\}$. The following result is known:

**Lemma 4.** [8]. If $f$ is nonconcave (convex) on $[a, b]$ then the following relation holds:

$$[u, v; f] \leq (<) [w, t; f], \ \forall v, w \in [u, t], v \neq w.$$  

An inequality analogous to (5) holds when $f$ is nonconvex (concave) on $[a, b]$.

2. The Convergence of the Steffensen Method

We shall consider that $f$ obeys the following hypotheses:

i. $f$ is continuous at $a$ and $b$;
ii. $f(a) \cdot f(b) < 0$;
iii. $f$ is increasing on $[a, b]$;
iv. $f$ is convex on $[a, b]$ and $f$ is continuous at $a$ and $b$;
v. $f$ is differentiable at $x^*$, the solution of (1), and $x^* \in (a, b)$.

**Remark 1.** Hypotheses iv. ensures the continuity of $f$ on $(a, b)$ (see, e.g. [3, p. 295]).

**Remark 2.** Hypotheses i.–iv. ensure the existence and the unicity of the solution $x^* \in (a, b)$ of equation (1).

Let $\alpha, \beta \in (a, b)$ be such that $f(\alpha) < 0$ and $f(\beta) > 0$ (their existence is ensured by hypotheses i.–iv.).
Consider the function $g : [\alpha, \beta] \to \mathbb{R}$ given by

$$g(x) = x - \frac{f(x)}{[a, \alpha; f]}$$

(6)

By iii. and iv. and Lemma 4 it follows that

$$[u, v; g] < 0, \quad \forall u, v \in (\alpha, \beta),$$

(7)

i.e., $g$ is decreasing.

We shall make the following hypotheses regarding the initial approximation $x_1$ in (3):

a) $f(x_1) < 0$;

b) $g(x_1) < \beta$.

Regarding the convergence of the Steffensen method (3) we prove the following result:

**Theorem 5.** Assume that $f$ obeys assumptions i.–v., that the function $g$ is given by (6) and $x_1$ obeys a) and b). Then the sequence $(x_n)_{n \geq 1}$ and $(g(x_n))_{n \geq 1}$ generated by (3) satisfy the following properties:

j. the sequence $(x_n)_{n \geq 1}$ is increasing and bounded;

jj. the sequence $(g(x_n))_{n \geq 1}$ is decreasing and bounded;

jjj. the following is true:

$$x_n < x^* < g(x_n), \quad \forall n \in \mathbb{N}.$$  

(8)

**Proof.** By (6) we get that $x^* = g(x^*)$. Since $x_1 < x^*$, and $g$ is decreasing, it follows $g(x_1) > g(x^*) = x^*$ and so $x_1 < x^* < g(x_1)$.

We show now that $x_2$ given by (3) verifies $x_1 < x_2 < x^*$. Since $f(x_1) < 0$ and $f$ is increasing, it follows $x_2 = x_1 - \frac{f(x_1)}{[x_1, g(x_1); f]} > x_1$. Further, it can be easily seen that the following identity holds:

$$x_1 - \frac{f(x_1)}{[x_1, g(x_1); f]} = g(x_1) - \frac{f(g(x_1))}{[x_1, g(x_1); f]},$$

whence, by (3) for $n = 1$ it follows $x_2 < g(x_1)$, since $f(g(x_1)) > 0$ and $[x_1, g(x_1); f] > 0$.

From the identity

$$f(x_2) = f(x_1) + [x_1, g(x_1); f](x_2 - x_1) + [x_2, x_1, g(x_1); f](x_2 - x_1)(x_2 - g(x_1))$$

taking into account (3) for $n = 1$ and the fact that $f$ is convex, we get $f(x_2) < 0$ and so $x_2 < x^*$.

By $x_2 > x_1$ it results $g(x_2) < g(x_1)$. We prove that $g(x_2) > x^*$. Since $x_2 < x^*$, from the monotony of $g$ it follows $g(x_2) > g(x^*) = x^*$. In conclusion, we get

$$x_1 < x_2 < x^* < g(x_2) < g(x_1).$$  

(9)
Assume now that for some \( n \geq 2 \), the elements obtained by (3) verify:

\[
 x_1 < x_2 < \cdots < x_n < \cdots < x^* < \cdots < g(x_n) < \cdots < g(x_2) < g(x_1). 
\]

Repeating the above reason for \( x_1 = x_n \) we get

\[
 x_n < x_{n+1} < x^* < g(x_{n+1}) < g(x_n). 
\]

From (10) and (11) one obtains the monotony of the sequences \((x_n)_{n \geq 1}\) and \((g(x_n))_{n \geq 1}\). Obviously, these sequence are bounded, so there exists \( \bar{x} = \lim_{n \to \infty} x_n \), and \( \lim_{n \to \infty} g(x_n) = g(\bar{x}) \), since \( g \) is continuous.

Passing to limit in (3) implies \( \bar{x} = \bar{x} - \frac{f(\bar{x})}{|x,g(\bar{x});f|} \) i.e. \( f(\bar{x}) = 0 \), and so \( \bar{x} = x^* \).

Relations (11) imply the following a posteriori errors

\[
 x^* - x_n \leq g(x_n) - x_n, \quad n = 1, 2, \ldots 
\]

Remark 3. Consider in (3) the function \( g : [\alpha, \beta] \to \mathbb{R} \),

\[
 g(x) = x - \frac{f(x)}{[\beta,b;f]} 
\]

If \( f \) is concave on \( [\alpha, \beta] \), then \([u,v;f] > [\beta,b;f], \forall u, v \in [\alpha, \beta]\) and so \( g \) is decreasing on \( [\alpha, \beta] \). Suppose now that hypotheses iv. and a) resp. b) imposed on \( f \) and \( g \) are replaced by

iv'. the function \( f \) is concave on \([a,b];\)

the initial value \( x_1 \) in (3) is such that

a'). \( f(x_1) > 0; \)

b'). \( g(x_1) > \alpha, \) with \( g \) given by (13).

Then the sequences \((x_n)_{n \geq 1}\) and \((g(x_n))_{n \geq 1}\) have the following properties:

j'. \( (x_n)_{n \geq 1} \) is decreasing;

jj'. \( (g(x_n))_{n \geq 1} \) is increasing;

jjj'. \( g(x_n) < x^* < x_n, \quad n = 1, 2, \ldots \)

The proof of these properties is similar to that given for Theorem 5. \( \square \)

REFERENCES


Received by the editors: March 12, 2003.