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AITKEN–STEFFENSEN TYPE METHODS FOR NONSMOOTH FUNCTIONS (III)*

ION PĂVĂLOIU†

Abstract. We provide sufficient conditions for the convergence of the Steffensen method for solving the scalar equation $f(x) = 0$, without assuming differentiability of f at other points than the solution x^*. We analyze the cases when the Steffensen method generates two sequences which approximate bilaterally the solution.
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1. INTRODUCTION

In this paper we consider the Steffensen method for approximating the solutions of the equations of the form

$$f(x) = 0 \tag{1}$$

with $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$. Let $g : [a, b] \to \mathbb{R}$ be such that the equation

$$x - g(x) = 0 \tag{2}$$

is equivalent to (1).

As it is well known, the Steffensen method consists in approximating the solution x^* of (1) by the sequence $(x_n)_{n\geq 1}$ given by

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]}, \quad n = 1, 2, \ldots, \ x_0 \in [a, b]. \tag{3}$$

We are interested in the following in the conditions under which the sequences $(x_n)_{n\geq 1}$ and $(g(x_n))_{n\geq 1}$ are monotone, and offer bilateral approximations to x^*. The importance of such sequences resides in the fact that at each iteration step we obtain a rigorous error bound. We shall construct the function g without assuming that f is differentiable on the whole interval $[a, b]$. In this sense, we shall use the divided differences of f.

Regarding the monotony and convexity of the function f we shall adopt the following definitions.

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†"T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68-1, 3400 Cluj-Napoca, Romania, e-mail: pavaloiu@ictp.acad.ro.

Definition 1. *The function f is nondecreasing (increasing) on $[a,b]$ if $[u,v;f] \geq 0$ (>0) $\forall u,v \in [a,b]$, while f is nonincreasing (decreasing) if $[u,v;f] \leq 0$ (<0) $\forall u,v \in [a,b]$.*

Definition 2. *The function f is nonconcave (convex) on $[a,b]$ if*

$$[u,v,w;f] \geq 0 \ (>0), \quad \forall u,v,w \in [a,b],$$

and is nonconvex (concave) if

$$[u,v,w;f] \leq 0 \ (<0), \quad \forall u,v,w \in [a,b].$$

Consider the function $p_{x_0} : [a,b] \setminus \{x_0\} \to \mathbb{R}$ given by

$$p_{x_0} = [x_0, x; f]\,. \tag{4}$$

Recall the following result:

Theorem 3. [3, p. 290].

a) *If f is nonconcave on $[a,b]$ then p_{x_0} is nondecreasing on $[a,b]$;*
b) *If f is convex on $[a,b]$ then p_{x_0} is increasing on $[a,b]$;*
c) *If f is nonconvex on $[a,b]$ then p_{x_0} is nonincreasing on $[a,b]$;*
d) *If f is concave on $[a,b]$ then p_{x_0} is decreasing on $[a,b]$.*

Consider now $u,v,w,t \in [a,b]$ such that $u \leq \min\{v,w,t\}$ and $t \geq \max\{u,v,w\}$. The following result is known:

Lemma 4. [8]. *If f is nonconcave (convex) on $[a,b]$ then the following relation holds:*

$$[u,v;f] \leq (<) \, [w,t;f]\,, \quad \forall v,w \in [u,t]\,,\ v \neq w. \tag{5}$$

An inequality analogous to (5) holds when f is nonconvex (concave) on $[a,b]$.

2. THE CONVERGENCE OF THE STEFFENSEN METHOD

We shall consider that f obeys the following hypotheses:

i. f is continuous at a and b;
ii. $f(a) \cdot f(b) < 0$;
iii. f is increasing on $[a,b]$;
iv. f is convex on $[a,b]$ and f is continuous at a and b;
v. f is differentiable at x^*, the solution of (1), and $x^* \in (a,b)$.

Remark 1. Hypotheses iv. ensures the continuity of f on (a,b) (see, e.g. [3, p. 295]). □

Remark 2. Hypotheses i.–iv. ensure the existence and the unicity of the solution $x^* \in (a,b)$ of equation (1). □

Let $\alpha, \beta \in (a,b)$ be such that $f(\alpha) < 0$ and $f(\beta) > 0$ (their existence is ensured by hypotheses i.–iv.).

Consider the function $g : [\alpha, \beta] \to \mathbb{R}$ given by

$$g(x) = x - \frac{f(x)}{[a, \alpha; f]} \tag{6}$$

By iii. and iv. and Lemma 4 it follows that

$$[u, v; g] < 0, \quad \forall u, v \in (\alpha, \beta), \tag{7}$$

i.e., g is decreasing.

We shall make the following hypotheses regarding the initial approximation x_1 in (3):

a) $f(x_1) < 0$;
b) $g(x_1) < \beta$.

Regarding the convergence of the Steffensen method (3) we prove the following result:

THEOREM 5. *Assume that f obeys assumptions* i.–v., *that the function g is given by* (6) *and x_1 obeys* a) *and* b). *Then the sequence $(x_n)_{n\geq 1}$ and $(g(x_n))_{n\geq 1}$ generated by* (3) *satisfy the following properties:*

j. *the sequence $(x_n)_{n\geq 1}$ is increasing and bounded;*
jj. *the sequence $(g(x_n))_{n\geq 1}$ is decreasing and bounded;*
jjj. *the following is true:*

$$x_n < x^* < g(x_n), \ \forall n \in \mathbb{N}. \tag{8}$$

Proof. By (6) we get that $x^* = g(x^*)$. Since $x_1 < x^*$, and g is decreasing, it follows $g(x_1) > g(x^*) = x^*$ and so $x_1 < x^* < g(x_1)$.

We show now that x_2 given by (3) verifies $x_1 < x_2 < x^*$. Since $f(x_1) < 0$ and f is increasing, it follows $x_2 = x_1 - \frac{f(x_1)}{[x_1, g(x_1); f]} > x_1$. Further, it can be easily seen that the following identity holds:

$$x_1 - \frac{f(x_1)}{[x_1, g(x_1); f]} = g(x_1) - \frac{f(g(x_1))}{[x_1, g(x_1); f]},$$

whence, by (3) for $n = 1$ it follows $x_2 < g(x_1)$, since $f(g(x_1)) > 0$ and $[x_1, g(x_1); f] > 0$.

From the identity

$$\begin{aligned} &f(x_2) = \\ &= f(x_1) + [x_1, g(x_1); f](x_2 - x_1) + [x_2, x_1, g(x_1); f](x_2 - x_1)(x_2 - g(x_1)) \end{aligned}$$

taking into account (3) for $n = 1$ and the fact that f is convex, we get $f(x_2) < 0$ and so $x_2 < x^*$.

By $x_2 > x_1$ it results $g(x_2) < g(x_1)$. We prove that $g(x_2) > x^*$. Since $x_2 < x^*$, from the monotony of g it follows $g(x_2) > g(x^*) = x^*$. In conclusion, we get

$$x_1 < x_2 < x^* < g(x_2) < g(x_1). \tag{9}$$

Assume now that for some $n \geq 2$, the elements obtained by (3) verify:

$$x_1 < x_2 < \cdots < x_n < \cdots < x^* < \cdots < g(x_n) < \cdots < g(x_2) < g(x_1). \tag{10}$$

Repeating the above reason for $x_1 = x_n$ we get

$$x_n < x_{n+1} < x^* < g(x_{n+1}) < g(x_n). \tag{11}$$

From (10) and (11) one obtains the monotony of the sequences $(x_n)_{n\geq 1}$ and $(g(x_n))_{n\geq 1}$. Obviously, these sequence are bounded, so there exists $\bar{x} = \lim_{n\to\infty} x_n$, and $\lim_{n\to\infty} g(x_n) = g(\bar{x})$, since g is continuous.

Passing to limit in (3) implies $\bar{x} = \bar{x} - \frac{f(\bar{x})}{[\bar{x},g(\bar{x});f]}$ i.e. $f(\bar{x}) = 0$, and so $\bar{x} = x^*$. □

Relations (11) imply the following a posteriori errors

$$x^* - x_n \leq g(x_n) - x_n, \quad n = 1, 2, \ldots \tag{12}$$

REMARK 3. Consider in (3) the function $g : [\alpha, \beta] \to \mathbb{R}$,

$$g(x) = x - \frac{f(x)}{[\beta, b; f]} \tag{13}$$

If f is concave on $[\alpha, \beta]$, then $[u, v; f] > [\beta, b; f]$, $\forall u, v \in [\alpha, \beta]$ and so g is decreasing on $[\alpha, \beta]$. Suppose now that hypotheses iv. and a) resp. b) imposed on f and g are replaced by

iv′. the function f is concave on $[a, b]$;

the initial value x_1 in (3) is such that

a′). $f(x_1) > 0$;

b′). $g(x_1) > \alpha$, with g given by (13).

Then the sequences $(x_n)_{n\geq 1}$ and $(g(x_n))_{n\geq 1}$ have the following properties:

j′. $(x_n)_{n\geq 1}$ is decreasing;

jj′. $(g(x_n))_{n\geq 1}$ is increasing;

jjj′. $g(x_n) < x^* < x_n, \quad n = 1, 2, \ldots$

The proof of these properties is similar to that given for Theorem 5. □

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