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ON CONVEXITY–LIKE INEQUALITIES (II)

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Abstract. We improve the classical Jensen inequality for convex functions by extending it to a wider class of functions. We also consider some weaker conditions for the weights occurring in this inequality.

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1. INTRODUCTION

One of the most important inequality in the theory of convex functions and many fields of mathematics connected with this theory is Jensen's inequality which states:

THEOREM 1. *If $f : I \to \mathbb{R}$ (I is an open interval in $\mathbb{R}$) is a convex function, $x_1, x_2, \ldots, x_n \in I$, $n \geq 2$, $p_1, p_2, \ldots, p_n > 0$, then*

$$f\Big(\tfrac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \leq \tfrac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{1}$$

where $P_n = \sum_{i=1}^{n} p_i$.

This inequality has an important role in mathematics and statistics and there exist a lot of its generalization, improvements and refinements. For example, it is reasonable to ask whether the condition $p_i > 0$ can be relaxed. Or, on the other hand, can we obtain inequality (8) for a wider class of functions? Some answers on the second question were given in [1] by Dragomirescu and Ivan, and in [3] by Pečarić and Pearce. Answer on the first question was given by Steffensen [8]. Namely, he obtained that the following theorem holds.

THEOREM 2. *If $f : I \to \mathbb{R}$ (I is an open interval in $\mathbb{R}$) is a convex function, $(x_1, x_2, \ldots, x_n) \in I^n$, $n \geq 2$, is a monotonic n-tuple, and if p_i are real numbers such that*

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$$(2) \qquad \bar{x} = \tfrac{1}{P_n} \sum_{i=1}^{n} p_i x_i \in I, \quad 0 \le P_k \le P_n, \quad k = 1, 2, \dots, n-1, \quad P_n > 0,$$

then (1) *holds.*

The previous theorem is known in literature under the name: Jensen–Steffensen's inequality. In this paper we show that the condition (2) can be relaxed. In Section 2 we suppose that f is a differentiable function and main result is a consequence of the generalized Steffensen's inequality [7]. In Section 3 we will pointed out that differentiability is not a necessary condition and we will be in position to prove a theorem about Jensen's type inequality directly, without using some other results. The proof is based on the identity which is first used in the proof of the Jensen-Steffensen inequality by J. Pečarić in [4].

2. CONSEQUENCE OF THE GENERALIZED STEFFENSEN INEQUALITY

We need the following Steffensen type inequality which is given in [7].

Here, we shall use these notation:

$$\mathbf{t} = (t_1, \dots, t_n), \quad [\mathbf{a}, \mathbf{b}] = \{\mathbf{t} : a_i \le t_i \le b_i, 1 \le i \le n\} = \prod_{i=1}^{n} [a_i, b_i].$$

If $\mathbf{t}$ and $\mathbf{x}$ are two n-tuples then by $\mathbf{t} + \mathbf{x}$ we will denote the n-tuple $(t_1 + x_1, \dots, t_n + x_n)$.

THEOREM 3. *Let μ be a measure such that $[\mathbf{a}, \mathbf{b}]$ is a finite μ-measurable set, f, g and fg be μ-integrable functions on $[\mathbf{a}, \mathbf{b}]$. Let $\Lambda = (\lambda_1, \dots \lambda_n)$ be a positive n-tuple, $[\mathbf{a}, \mathbf{a} + \lambda] \subset [\mathbf{a}, \mathbf{b}]$ and A be a real number such that*

$$A\mu([\mathbf{a}, \mathbf{a} + \Lambda]) = \int_{[\mathbf{a}, \mathbf{b}]} g \mathrm{d}\mu,$$

and let one of the following cases be satisfied μ-almost everywhere:

1. $g(\mathbf{t}) \le A$ *and* $f(\mathbf{t}) \ge f(\mathbf{a} + \Lambda)$, *for* $\mathbf{t} \in [\mathbf{a}, \mathbf{a} + \Lambda]$;
   $g(\mathbf{t}) \ge 0$ *and* $f(\mathbf{t}) \le f(\mathbf{a} + \Lambda)$, *for* $\mathbf{t} \in [\mathbf{a}, \mathbf{b}] \backslash [\mathbf{a}, \mathbf{a} + \Lambda]$;
2. $g(\mathbf{t}) \ge A$ *and* $f(\mathbf{t}) \le f(\mathbf{a} + \Lambda)$, *for* $\mathbf{t} \in [\mathbf{a}, \mathbf{a} + \Lambda]$;
   $g(\mathbf{t}) \le 0$ *and* $f(\mathbf{t}) \ge f(\mathbf{a} + \Lambda)$, *for* $\mathbf{t} \in [\mathbf{a}, \mathbf{b}] \backslash [\mathbf{a}, \mathbf{a} + \Lambda]$.

Then

$$A \int_{[\mathbf{a}, \mathbf{a} + \Lambda]} f \mathrm{d}\mu \ge \int_{[\mathbf{a}, \mathbf{b}]} fg \mathrm{d}\mu.$$

This is a direct multidimensional generalization of the right-hand side of Steffensen's inequality which states that if f is a nonincreasing function on $[a, b]$ and g is integrable with $g(t) \in [0, 1]$, then the following inequalities hold

$$(3)\qquad \int_{b-\lambda}^{b} f(t)\mathrm{d}t \le \int_a^b f(t)g(t)\mathrm{d}t \le \int_a^{a+\lambda} f(t)\mathrm{d}t, \qquad \lambda = \int_a^b g(t)\mathrm{d}t.$$

Generalization of the left-hand side of the inequality is given in the following theorem.

THEOREM 4. *Let μ be a measure such that $[\mathbf{a},\mathbf{b}]$ is a finite μ-measurable set, f, g and fg be μ-integrable functions on $[\mathbf{a},\mathbf{b}]$. Let $\Lambda = (\lambda_1,\dots,\lambda_n)$ be a positive n-tuple, $[\mathbf{b}-\Lambda,\mathbf{b}] \subset [\mathbf{a},\mathbf{b}]$ and A be a real number such that*

$$A\mu([\mathbf{b}-\Lambda,\mathbf{b}]) = \int_{[\mathbf{a},\mathbf{b}]} g\mathrm{d}\mu,$$

and let one of the following three cases be satisfied μ-almost everywhere:

1. $g(\mathbf{t}) \le A$ *and* $f(\mathbf{t}) \le f(\mathbf{b}-\Lambda)$, *for* $\mathbf{t} \in [\mathbf{b}-\Lambda,\mathbf{b}]$;
   $g(\mathbf{t}) \ge 0$ *and* $f(\mathbf{t}) \ge f(\mathbf{b}-\Lambda)$, *for* $\mathbf{t} \in [\mathbf{a},\mathbf{b}]\backslash[\mathbf{b}-\Lambda,\mathbf{b}]$;
2. $g(\mathbf{t}) \ge A$ *and* $f(\mathbf{t}) \ge f(\mathbf{b}-\Lambda)$, *for* $\mathbf{t} \in [\mathbf{b}-\Lambda,\mathbf{b}]$;
   $g(\mathbf{t}) \le 0$ *and* $f(\mathbf{t}) \le f(\mathbf{b}-\Lambda)$, *for* $\mathbf{t} \in [\mathbf{a},\mathbf{b}]\backslash[\mathbf{b}-\Lambda,\mathbf{b}]$.

Then

$$A\int_{[\mathbf{b}-\Lambda,\mathbf{b}]} f\mathrm{d}\mu \le \int_{[\mathbf{a},\mathbf{b}]} fg\mathrm{d}\mu.$$

Proofs of the above mentioned theorems are given in [7]. Now, we are in the situation to pointed out the following result which improve classical Jensen's inequality.

THEOREM 5. *Let $f:(a,b)\to\mathbb{R}$ be a differentiable function, $(x_1,x_2,\dots,x_n) \in (a,b)^n$ be a nonincreasing n-tuple, and $p_1,\dots,p_n$ be a real numbers such that there exists integer $k \in \{1,2,\dots,n-1\}$ with properties*

$$(4)\qquad \tfrac{1}{P_n}\sum_{i=1}^n p_i x_i \quad\in\quad (x_{k+1},x_k]$$

$$\begin{aligned} P_j &\ge 0, \quad j=1,2,\dots,k;\\ (5)\qquad \bar{P}_j &\ge 0 \quad j=k+1,\dots,n, \quad P_n>0, \end{aligned}$$

where $P_j = \sum_{i=1}^{j} p_i$ and $\bar{P}_j = \sum_{i=j}^{n} p_i$.

If f has property that

$$\begin{aligned} f'(x) &\le f'\Big(\tfrac{1}{P_n}\sum_{i=1}^n p_i x_i\Big), \quad \text{for } x \le \tfrac{1}{P_n}\sum_{i=1}^n p_i x_i \text{ and}\\ (6)\qquad f'(x) &\ge f'\Big(\tfrac{1}{P_n}\sum_{i=1}^n p_i x_i\Big), \quad \text{for } x \ge \tfrac{1}{P_n}\sum_{i=1}^n p_i x_i, \end{aligned}$$

then (1) *holds.*

Proof. We use idea from [5]. Setting $g(t)=g_j$ on $(x_{j+1},x_j]$, $j=1,\dots,n-1$, where

$$g_j = \tfrac{P_j}{P_n}, \quad\text{and}\quad g(x_n)=g_n=1,$$

we easily check that the following holds:

$$g_j \leq 1 \quad \text{for} \quad j = n, \ldots, k,$$
$$g_j \geq 0 \quad \text{for} \quad j = 1, \ldots, k$$

and after some simple calculations we get

$$\lambda = \int_{x_n}^{x_1} g(t)\mathrm{d}t = \tfrac{1}{P_n}\sum_{i=1}^{n} p_i x_i - x_n.$$

Functions g and $-f'$ satisfy assumptions from Theorem 3, case 1, for $n = 1$, so, the following inequality holds

$$\int_{x_n}^{x_1} f'(t)g(t)\mathrm{d}t \geq \int_{x_n}^{x_n+\lambda} f'(t)\mathrm{d}t$$

and we get

$$\sum_{k=1}^{n-1} (f(x_k) - f(x_{k+1}))g_k \geq f(x_n + \lambda) - f(x_n)$$

wherefrom it can be obtained (1). □

Furthermore, Theorem 5 is also true if $(x_1, \ldots, x_n)$ is a nondecreasing sequence; the only change needed in the hypothesis is to require that

$$\tfrac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in (x_k, x_{k+1}].$$

In this case, the proof is based on the inequality described in Theorem 4.

Once more, we stress that under the conditions of the previous Theorem 5, the Jensen inequality holds not only for convex functions but also for the wider class of functions.

3. FURTHER RESULT

Let $c \in (a, b)$ and λ be a real number. We will say that $f : (a, b) \to \mathbb{R}$ has property K1(c,λ), if for any $z, y \in (a, b), z, y \geq c$,

$$\frac{f(z) - f(y)}{z - y} \geq \lambda \tag{7}$$

holds.

If for any $z, y \in (a, b), z, y \leq c$ inequality (7) holds then we will say that f has property K2(c,λ).

If in (7) the reversed inequality holds, then we will say that f has property RK1(c,λ) or RK2(c,λ) respectively.

If f is differentiable function on (a, b) and if f has a property K1(c,λ) for some $c \in (a, b)$, then for f' the following holds

$$f'(x) \geq \lambda, \quad \forall x \geq c.$$

Obviously, if f is a nondecreasing function on (a, b) then f has properties K1(c,0) and K2(c,0) for any $c \in (a, b)$. Also, it is known that a convex function

f defined on an interval (a,b) has properties $\mathrm{K1}(c, f'_+(c))$ and $\mathrm{RK2}(c, f'_-(c))$ for any $c \in (a,b)$.

THEOREM 6. *Let $(x_1,\dots,x_n) \in (a,b)^n$ be a nonincreasing sequence and $p_1,\dots,p_n$ be real numbers such that $P_n = \sum\limits_{i=1}^{n} p_i > 0$. Let us suppose that there exists an integer $m \in \{1,2,\dots,n-1\}$ such that $\bar{x} = \frac{1}{P_n}\sum\limits_{i=1}^{n} p_i x_i \in (x_{m+1}, x_m]$.*

1. *If $P_i \geq 0$, for $i = 1,\dots,m$, $\bar{P}_i \geq 0$, for $i = m,\dots,n$ and if $f : (a,b) \to \mathbb{R}$ has properties $\mathrm{K1}(\bar{x},\lambda)$ and $\mathrm{RK2}(\bar{x},\lambda)$ then*

$$f\Big(\tfrac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \leq \tfrac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{8}$$

   holds, where $P_i = \sum\limits_{j=1}^{i} p_j$ and $\bar{P}_i = \sum\limits_{j=i}^{n} p_j$.
2. *If $P_i \geq 0$, for $i = 1,\dots,m$, $\bar{P}_i \leq 0$, for $i = m,\dots,n$ and if $f : (a,b) \to \mathbb{R}$ has properties $\mathrm{K1}(\bar{x},\lambda)$ and $\mathrm{K2}(\bar{x},\lambda)$ then (8) holds.*
3. *Similarly, if $P_i \leq 0$, for $i = 1,\dots,m$, $\bar{P}_i \geq 0$, for $i = m,\dots,n$ and if $f : (a,b) \to \mathbb{R}$ has properties $\mathrm{RK1}(\bar{x},\lambda)$ and $\mathrm{RK2}(\bar{x},\lambda)$ then (8) holds.*

Proof. The proof is based on the following identity [4]:

$$\begin{aligned}
f(\bar{x}) - \tfrac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) = &\sum_{i=1}^{m-1} \big[\lambda(x_i - x_{i+1}) - \big(f(x_i) - f(x_{i+1})\big)\big]\tfrac{P_i}{P_n} \\
&+ \big[\lambda(x_m - \bar{x}) - \big(f(x_m) - f(\bar{x})\big)\big]\tfrac{P_m}{P_n} \\
&+ \big[f(\bar{x}) - f(x_{m+1}) - \lambda(\bar{x} - x_{m+1})\big]\tfrac{\bar{P}_{m+1}}{P_n} \\
&+ \sum_{i=m+1}^{n-1} \big[f(x_i) - f(x_{i+1}) - \lambda(x_i - x_{i+1})\big]\tfrac{\bar{P}_{i+1}}{P_n},
\end{aligned}$$

where $P_i = \sum\limits_{j=1}^{i} p_j$ and $\bar{P}_i = \sum\limits_{j=i}^{n} p_j$.

Let us suppose that assumptions from case (1) hold. Then the right-hand side of the previous identity is non-positive, so the left-hand side is also non-positive, and inequality (8) is proved. Other cases can be proved on the same way. □

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