A CONVERGENCE ANALYSIS OF AN ITERATIVE ALGORITHM OF ORDER 1.839... UNDER WEAK ASSUMPTIONS

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Abstract. We provide new and weaker sufficient local and semilocal conditions for the convergence of a certain iterative method of order 1.839... to a solution of an equation in a Banach space. The new idea is to use center-Lipschitz/Lipschitz conditions instead of just Lipschitz conditions on the divided differences of the operator involved. This way we obtain finer error bounds and a better information on the location of the solution under weaker assumptions than before.


Keywords. Banach space, majorizing sequence, Halley method, Euler–Chebyshev method, divided differences of order one and two, Fréchet-derivative, $R$-order of convergence, convergence radius.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution $x^*$ of equation

$$F(x) = 0,$$

where $F$ is a Fréchet differentiable operator on an open convex subset $D$ of a Banach space $X$ with values in a Banach space $Y$.

The iterations

$$x_{n+1} = x_n - L_n^{-1}F(x_n),$$

$$L_n = [x_n, x_{n-1}] + [x_{n-2}, x_n] - [x_{n-2}, x_{n-1}], \quad n \geq 0,$$

have been already used to generate a sequence converging to $x^*$ with $R$-order 1.839... [4], [12] and [13]. Here, $[x, y] \in L(X, Y)$, $[x, y, z] \in L(X, L(X, Y))$ denote divided differences of order one and two respectively of operator $F$ satisfying

$$[x, y](y - x) = F(y) - F(x)$$

and

$$[x, y, z](y - z) = [x, y] - [x, z]$$

for all $x, y, z \in D$ [4].
Method (2) is considered to be a discretized version of the famous cubically convergent methods of Euler–Chebyshev (tangent hyperbola) or Halley (parabola) \[1\]–\[3\], \[6\], \[10\] and \[11\]. Discretized versions of the above methods using divided differences of order one and two or just one have also been considered in \[5\], \[7\] and \[9\].

Here we provide a new local and semilocal convergence analysis for method (2). Using Lipschitz and center Lipschitz conditions on the divided differences of operator $F$ instead of just Lipschitz conditions we introduce weaker sufficient convergence conditions than before. Moreover we obtain finer error bounds on the distances involved as well as a better information on the location of the solution $x^\ast$. Furthermore in the case of local analysis a larger convergence radius is obtained.

2. SEMILOCAL ANALYSIS

Let $d_i$, $i = 0, 1, \ldots, 5$, $\eta_0$, $\eta_1$ be non-negative parameters and $\delta \in [0, 1)$. Define the parameters $\alpha_0$, $\alpha_1$, $\alpha_2$, $\beta_0$, $\beta_1$ by

\[
\begin{align*}
\alpha_2 &= (1 - \delta^2)d_5, \\
\alpha_1 &= \delta\{\delta(d_1 + \delta d_0) - (1 - \delta^2)(d_3 + \delta d_2)\}, \\
\alpha_0 &= -\delta(1 - \delta)(\eta_0 + \eta_1)d_5\eta_1, \\
\beta_1 &= d_3 + \delta(d_0 + d_2), \\
\beta_0 &= (\eta_0 + \eta_1)\eta_1(d_5 + \delta d_4) - \delta,
\end{align*}
\]

and functions $f$, $g$ by

\[
\begin{align*}
f(t) &= \alpha_2 t^2 + \alpha_1 t + \alpha_0, \\
g(t) &= \beta_1 t + \beta_0.
\end{align*}
\]

We can show the following result on majorizing sequences.

**Theorem 1.** Let $\eta$ be a non-negative parameter such that

\[
\eta \leq \begin{cases} 
\min\{\alpha_3, \beta_2\}, & \text{if } \beta_1 \neq 0, \\
0, & \text{if } \alpha_0 = 0, \\
\alpha_3, & \text{if } \beta_1 = 0,
\end{cases}
\]

provided

\[
\beta_0 \leq 0,
\]

where $\alpha_3$, $\beta_2$ are the non-negative zeros of functions $f$ and $g$ respectively. Then
(a) Iteration \( \{t_n\} \), \( n \geq -2 \), given by
\[
\begin{align*}
t_{-2} &= 0, \\
t_{-1} &= \eta_0, \\
t_0 &= \eta_0 + \eta_1, \\
t_1 &= \eta_0 + \eta_1 + \eta, \\
t_{n+2} &= t_{n+1} + \frac{d_3(t_{n+1} - t_n) + d_2(t_n - t_{n-2}) + d_1(t_{n-2} - t_{n-3})}{1 - d_4 \eta_1 (\eta_0 + \eta_1) - d_3 (t_{n+1} - t_0) - d_2(t_n - t_{n-2}) - d_1(t_{n-2} - t_{n-3}) - d_0(t_{n-3} - t_{n-4})} (t_{n+1} - t_n),
\end{align*}
\]
\( n \geq 0, \)
is non-decreasing, bounded above by
\[
t^{**} = \frac{\eta}{1 - \delta} + \eta_0 + \eta_1
\]
and converges to \( t^* \) such that
\[
0 \leq t^* \leq t^{**}.
\]
Moreover, the following error bounds hold for all \( n \geq 0 \)
\[
0 \leq t_{n+2} - t_{n+1} \leq \delta(t_{n+1} - t_n) \leq \delta^{n+1} \eta.
\]
(b) Iteration \( \{s_n\} \), \( n \geq -2 \), given by
\[
\begin{align*}
s_{-2} - s_{-1} &= \eta_1, \\
s_{-1} - s_0 &= \eta_0, \\
s_0 - s_1 &= \eta, \\
s_{n+2} - s_{n+1} &= \frac{d_3(s_{n+1} - s_{n+2}) + d_2(s_{n+2} - s_{n+3}) + d_1(s_{n+3} - s_{n+4})}{1 - d_4 \eta_1 (\eta_0 + \eta_1) - d_3 (s_{n+1} - s_{n+2}) - d_2(s_{n+2} - s_{n+3}) - d_1(s_{n+3} - s_{n+4}) - d_0(s_{n+4} - s_{n+5})} (s_n - s_{n+1}),
\end{align*}
\]
\( n \geq 0, \)
for \( s_{-1}, s_0, s_1 \geq 0 \) is non-increasing, bounded below by
\[
s^{**} = s_0 - \frac{\eta}{1 - \delta}
\]
and converges to some \( s^* \) such that
\[
0 \leq s^{**} \leq s^*.
\]
Moreover, the following error bounds hold for all \( n \geq 0 \):
\[
0 \leq s_{n+1} - s_{n+2} \leq \delta(s_n - s_{n+1}) \leq \delta^{n+1} \eta.
\]
\[\text{Proof.}\] (a) We must show:
\[
(d_3 + \delta d_2)(t_{k+1} - t_k) + d_5(t_k - t_{k-2})(t_{k-1} - t_{k-2}) + d_0(t_{k+1} - t_0) + \delta d_4(t_k - t_0) + \eta_1(\eta_0 + \eta_1) \leq \delta
\]
\[
1 - d_4 \eta_1 (\eta_0 + \eta_1) - d_3 t_{k+1} + d_2(t_{k+1} - t_k) > 0
\]
for all \( k \geq 0. \)
Inequalities (22) and (23) hold for \( k = 0 \) by the initial conditions. But then (14) gives
\[
0 \leq t_2 - t_1 \leq \delta(t_1 - t_0) .
\]
Let us assume (22), (23) and (17) hold for all \( k \leq n + 1 \). By the induction hypotheses we get
\[
(d_3 + \delta d_2)(t_{k+2} - t_{k+1}) + d_5(t_{k+1} - t_{k+1})(t_{k+1} - t_k) + \delta d_0(t_{k+2} - t_0) + \delta d_1(t_{k+1} - t_0) + \delta d_4\eta_1(\eta_0 + \eta_1) \leq
\]
\[
(3 + \delta d_2)\delta^{k+1} + d_5(\delta^k + \delta^{k-1})\eta^2 \delta^k + \delta d_0 \frac{\eta}{1 - \delta} + \delta d_1 \frac{\eta}{1 - \delta} + \delta d_4\eta_1(\eta_0 + \eta_1).
\]
It is clear that (25) will be bounded above by \( \delta \) if
\[
(d_3 + \delta d_2)\delta^{k+1} + d_5(\delta^k + \delta^{k-1})\eta^2 \delta^k + \delta d_0 \frac{\eta}{1 - \delta} + \delta d_1 \frac{\eta}{1 - \delta} + \delta d_4\eta_1(\eta_0 + \eta_1) \leq
\]
\[
(3 + \delta d_2)\eta + d_5(\eta_0 + \eta_1)\eta_1 + \delta d_0\eta + \delta d_4\eta_1(\eta_0 + \eta_1)
\]
or
\[
(3 + \delta d_2)(1 - \delta)\delta^{k+1} + d_5(1 - \delta^2)\eta^2 \delta^k + \delta d_0\eta + \delta d_1\eta \leq
\]
\[
(3 + \delta d_2)(1 - \delta)(3 + \delta d_2)\eta + (1 - \delta)d_5(\eta_0 + \eta_1)\eta_1 + (1 - \delta)\delta d_0\eta
\]
or, for \( k \geq 0 \),
\[
(3 + \delta d_2)(1 - \delta)\eta + d_5(1 - \delta)(\eta_0 + \eta_1)\eta_1 + (1 - \delta)\delta d_0\eta
\]
or, for \( \delta \neq 0 \),
\[
\alpha_2\eta^2 + \alpha_1\eta + \alpha_0 \leq 0 ,
\]
which is true by the choice of \( \eta \). By the same proof as above we show (23) for \( k = n + 1 \).

We must also show:
\[
t_k \leq t^{**}, \quad k \geq 1 .
\]
For \( k = 1,2 \) we have \( t_1 \leq t^* \) and \( t_2 \leq t_1 + \delta\eta = \eta_0 + \eta_1 + (1 + \delta)\eta \leq t^{**} \). Assume (27) holds for all \( k \leq n + 1 \). It follows from (17) that
\[
t_{k+2} \leq t_{k+1} + \delta(t_{k+1} - t_k) \leq t_1 + \delta(t_1 - t_0) + \delta(t_{k+1} - t_k) \]
\[
\leq t_1 + \delta(t_1 - t_0) + \cdots + \delta(t_{k+1} - t_k) \leq \eta_0 + \eta_1 + \delta\eta + \cdots + \delta^{k+1} \eta
\]
\[
= \eta_0 + \eta_1 + \frac{1 - \delta^{k+2}}{1 - \delta} \eta
\]
\[
< t^{**} .
\]
That is, \( \{ t_n \}, \quad n \geq 0 \), is bounded above \( t^{**} \). By (22) and (23) we get
\[
t_{k+2} - t_{k+1} \geq 0 .
\]
Hence sequence \{t_n\}, \(n \geq 0\), is also non-decreasing and as such it converges to some \(t^*\) satisfying (16).

(b) The proof follows along the lines of part (a).

That completes the proof of Theorem 1. \(\square\)

We show the following semilocal convergence theorem for method (2).

**Theorem 2.** Let \(F : D \subseteq X \to Y\) be a differentiable operator with divided differences of the first and second order denoted by \([\cdot, \cdot], [\cdot, \cdot, \cdot]\) respectively. Assume:

- there exist points \(x_{-2}, x_{-1}, x_0 \in D\) so \(L_0\) is invertible and non-negative numbers \(\eta, d_i, i = 0, 1, \ldots, 5\) such that

\[
\begin{align*}
\|L_0^{-1}(x_0, x_0) - [y, y]\| &\leq d_0\|x_0 - y\|, \\
\|L_0^{-1}(x, x_0) - [x, y]\| &\leq d_1\|x_0 - y\|, \\
\|L_0^{-1}(x_0, y) - [x, z]\| &\leq d_2\|y - x\|, \\
\|L_0^{-1}(x, y) - [x, x]\| &\leq d_3\|y - x\|, \\
\|L_0^{-1}(x, y, x_0) - [z, y, x_0]\| &\leq d_4\|y - x\|, \\
\|L_0^{-1}(x, y, y) - [z, z, y]\| &\leq d_5\|x - y\|, \quad \text{for all } x, y, z \in D, \\
\|x_{n-1} - x_0\| &\leq \eta_1, \quad \|x_{n-1} - x_{-2}\| \leq \eta_0, \quad \|x_1 - x_0\| \leq \eta;
\end{align*}
\]

- hypotheses of Theorem 1 hold, and

\[
\mathcal{U}(x_0, t^*) = \{ x \in X : \|x - x_0\| \leq t^* \} \subseteq D.
\]

Then method \(\{x_n\}\), \(n \geq 0\), generated by (2) is well defined, remains in \(\mathcal{U}(x_0, t^*)\) for all \(n \geq 0\) and converges to a solution \(x^* \in \mathcal{U}(x_0, t^*)\) of equation \(F(x) = 0\). Moreover, the following error bounds hold for all \(n \geq 0\):

\[
\|x_{n+2} - x_{n+1}\| \leq t_{n+2} - t_{n+1}
\]

and

\[
\|x_n - x^*\| \leq t^* - t_n.
\]

Furthermore, if there exists \(R \geq t^*\) such that

\[
\mathcal{U}(x_0, R) \subseteq D,
\]

\[
\|L_0^{-1}(x, y) - [z, w]\| \leq d_6(\|x - z\| + \|y - w\|), \quad \text{for all } x, y, z, w \in D,
\]

and

\[
d_6(R + t^* + 2\eta_0 + 2\eta_1) \leq 1
\]

or \([\cdot, \cdot]\) is symmetric and

\[
d_6(R + t^* + \eta_0 + \eta_1) + d_1(\eta_0 + \eta_1) \leq 1,
\]

the solution \(x^*\) is unique in \(\mathcal{U}(x_0, R)\).
Proof. Let us prove
\[(43) \quad \|x_{k+1} - x_k\| \leq t_{k+1} - t_k \quad k \geq -2.\]
For \(k = -2, -1, 0\) (43) holds by the initial conditions. Assume (43) holds for all \(n \leq k\). Using (29), (30), (31), (33) and (43) we obtain
\[
\begin{align*}
\|L_0^{-1}(L_0 - L_{k+1})\| &= \|L_0^{-1}(L_0 - [x_0, x_0] + [x_0, x_0] - [x_{k+1}, x_0] + [x_{k+1}, x_0] - [x_{k+1}, x_k] - [x_{k+1}, x_k] - L_{k+1})\| \\
&\leq \|L_0^{-1}([x_0, x_0] - [x_{k+1}, x_0])\| + \|L_0^{-1}([x_{k+1}, x_0] - [x_{k+1}, x_k])\| \\
&\leq d_4\|x_{k+1} - x_0\|\|x_0 - x_0\| + d_0\|x_{k+1} - x_0\| \\
&\quad + d_1\|x_{k+1} - x_0\| + d_2\|x_{k+1} - x_{k+1}\| \\
&\leq d_4\eta_0(\eta_0 + \eta_1) + d_0(t_{k+1} - t_0) + d(t_k - t_0) + d_2(t_{k+1} - t_k) \\
&\leq d_4\eta_0(\eta_0 + \eta_1) + d_0(t^* - t_0) + d_1(t^* - t_0) + d_2\delta_k\eta \\
&< 1.
\end{align*}
\[(44) \quad \|L_0^{-1}L_0\| \leq 1 - d_4\eta_0(\eta_0 + \eta_1) - d_0(t_{k+1} - t_0) - d_1(t_k - t_0) - d_2(t_{k+1} - t_k)\]^{-1}.
Moreover, we have
\[
\begin{align*}
\|L_0^{-1}([x_k, x_{k+1}] - L_k)\| &= \|L_0^{-1}([x_k, x_{k+1}] - [x_k, x_k] + [x_k, x_k] - L_k)\| \\
&\leq \|L_0^{-1}([x_k, x_{k+1}] - [x_k, x_k])\| + \|L_0^{-1}([x_k, x_{k+1}] - [x_k, x_{k+1}])(x_k - x_{k+1})\| \\
&\leq d_3\|x_{k+1} - x_k\| + d_5\|x_k - x_{k-2}\|\|x_k - x_{k-1}\| \\
&\leq d_3(t_{k+1} - t_k) + d_5(t_k - t_{k-2})(t_k - t_{k-1}).
\end{align*}
\[(46) \quad \|x_{k+2} - x_{k+1}\| \leq \|L_{k+1}^{-1}L_0\|\|L_0^{-1}([x_k, x_{k+1}] - L_k)\|\|x_{k+1} - x_k\| \leq t_{k+2} - t_{k+1},\]
which shows (43).
Theorem 1 and (47) imply \( \{x_n\}, n \geq 0 \), is a Cauchy sequence in a Banach space \( X \) and as such it converges to some \( x^* \in \overline{U}(x_0, t^*) \), since \( \overline{U}(x_0, t^*) \) is a closed set.

By (43) we get
\[
\|x_n - x_m\| \leq t_m - t_n, \quad -2 \leq n \leq m,
\]
while by letting \( m \to \infty \) in (48) we obtain (38).

Finally, by letting \( k \to \infty \) in (47), we get
\[
F(x^*) = 0.
\]
To show uniqueness, let \( y^* \in U(x_0, R) \) be a solution of equation \( F(x) = 0 \). We can have in turn
\[
\frac{L_0^{-1}([y^*, x^*] - L_0)}{L_0^{-1}([y^*, x^*] - [x_0, x_2])} \leq d_6 R + t^* + 2(\eta_0 + \eta_1) + d_1 (\eta_0 + \eta_1) \leq 1.
\]

It follows by (49) and the Banach Lemma on invertible operators that \([y^*, x^*]\) is invertible. Hence, from
\[
F(x^*) - F(y^*) = [y^*, x^*](x^* - y^*),
\]
we deduce that \( x^* = y^* \).

If \([\cdot, \cdot]\) is symmetric as in (49) we get
\[
\frac{L_0^{-1}([y^*, x^*] - L_0)}{L_0^{-1}([y^*, x^*] - [x_0, x_2])} \leq d_6 R + t^* + 2(\eta_0 + \eta_1) + d_1 (\eta_0 + \eta_1) \leq 1.
\]
We conclude again that \( x^* = y^* \).

That completes the proof of Theorem 2. \( \square \)

The proof of the following result follows exactly as in Theorem 2 but using part (b) of Theorem 1.

**Theorem 3.** Assume hypotheses of Theorems 1 and 2 hold.
Then method \( \{x_n\}, n \geq 0 \), generated by (2) is well defined, remains in \( U(x_0, s^*) \) for all \( n \geq 0 \) and converges to a solution \( x^* \in U(x_0, s^*) \) of equation \( F(x) = 0 \). Moreover, the following error bounds hold for all \( n \geq 0 \):
\[
\|x_{n+2} - x_{n+1}\| \leq s_{n+1} - s_{n+2}
\]
and
\[
\|x_n - x^*\| \leq s_n - s^*.
\]
Furthermore if there exists \( R_1 \geq s^* \) such that, together with (40),
\[
U(x_0, R_1) \subseteq D
\]
holds, and
\[
d_6 [R_1 + s^* + 2(\eta_0 + R_1)] \leq 1
\]
or

\[ [\cdot, \cdot] \text{ is symmetric} \]

and

\[ d_6(R_1 + s^* + \eta_0 + \eta_1) + d_1(\eta_0 + \eta_1) \leq 1, \]

the solution \( x^* \) is unique in \( U(x_0, R) \).

**Remark 1.** In [12, Th. 5.1], condition (40) was used together with

\[ \| L_0^{-1}(x, y, z) - (u, y, z) \| \leq d_7 \| x - u \| \]

for all \( x, y, z, v \in D \) to show convergence of method (2).

The following error bounds were found

\[ \| x_{n+1} - x_n \| \leq v_n - v_{n+1} \]

and

\[ \| x_n - x^* \| \leq v_n - v^*, \]

where,

\[ v^* = \lim_{n \to \infty} v_n, \]

and \( \{v_n\} \) is similar to \( \{s_n\} \) but using \( d_6, d_7, \eta_0, \eta_1, \eta \) instead of \( d_0, d_1, d_2, d_3, d_4, d_5, \eta_0, \eta_1, \eta \). Note also that in general

\[ d_0 \leq d_1 \leq d_3 \leq d_2 \leq d_6 \]

and

\[ d_4 \leq d_5 \leq d_7. \]

Hence we can easily obtain by induction

\[ s_n - s_{n+1} \leq v_n - v_{n+1} \]

and

\[ s_n - s^* \leq v_n - v^*. \]

That is, under weaker convergence conditions we obtain finer error bounds. □

3. LOCAL ANALYSIS

We can show the following local results for method (2).
Theorem 4. Let $F : D \subseteq X \to Y$ be a differentiable operator. Assume $F$ has divided differences of the first and second order such that:

\[(65)\quad F'(x^*) = [x^*, x^*],\]

\[(66)\quad \|F'(x^*)^{-1}([x^*, x^*] - [x, x^*])\| \leq a_0\|x - x^*\|,\]

\[(67)\quad \|F'(x^*)^{-1}([x, x^*] - [x, y])\| \leq b_0\|x - x^*\|,\]

\[(68)\quad \|F'(x^*)^{-1}([x, x^*] - [z, x^*])\| \leq c_0\|z - x^*\|,\]

\[(69)\quad \|U(x_0)\| \leq c\|u - v\|,\]

\[(70)\quad \mathcal{U}(x^*, r^*) \subseteq D,\]

for all $x, y, z, u, v \in D$, where $x^*$ is a simple zero of $F$, and

\[(71)\quad r^* = \frac{2}{a_0 + 2b_0 + \sqrt{(a_0 + 2b_0)^2 + 8(c + c_0)}}.\]

Then method (2) is well defined, remains in $U(x^*, r^*)$ for all $n \geq 0$, and converges to $x^*$ provided that $x_{-1}, x_{-2}, x_0 \in U(x^*, r^*)$.

Moreover, the following error bounds hold for all $n \geq 0$:

\[(72)\quad \|x_{n+1} - x^*\| \leq \frac{b_0\|x_n - x^*\| + c([\|x_{n-1} - x^*\| + \|x_{n-2} - x^*\|])}{1 - (a_0 + b_0)\|x_n - x^*\| - c_0([\|x_n - x^*\| + \|x_{n-2} - x^*\|])\|x_{n-1} - x^*\|} \|x_n - x^*\| = \alpha_n.\]

Proof. We first show linear operator

\[L \equiv L(x, y, z) = [x, y] + [z, x] - [z, y], \quad x, y, z \in U(x^*, r^*)\]

is invertible. By (65), (67), (69), we get in turn

\[(73)\quad \|F'(x^*)^{-1}(F'(x^*) - L)\| = \]

\[= \|F'(x^*)^{-1}([x^*, x^*] - [x, x^*] + [z, x^*] - [z, x] + [x, x^*] - [x, y] - [z, x^*] + [z, y])\| \]

\[\leq \|F'(x^*)^{-1}([x^*, x^*] - [x, x^*])\| + \|F'(x^*)^{-1}([z, x^*] - [z, x])\| + \|F'(x^*)^{-1}([x, x^*] - [x, y] - [z, x^*] + [z, y])\| \]

\[\leq (a_0 + b_0)\|x - x^*\| + c\|x - z\| \cdot \|x^* - y\| \]

\[\leq (a_0 + b_0)\|x - x^*\| + c(\|x - x^*\| + \|z - x^*\|)\|y - x^*\| \]

\[< (a_0 + b_0)r^* + 2c(r^*)^2 \]

by the choice of $r^*$. It follows from (73) and the Banach Lemma on invertible operators that $L$ is invertible.
By (2) we can write

\[
(75) \quad x, y, u, v
\]

Estimate (72) now follows from (74), (75) and (76). By the choice of (76) we get

\[
(79) \quad \text{and from which it follows}
\]

Moreover, by (67) and (69) we get

\[
\|F'(x^*)^{-1}([x_k, x^*] - L_k)\| = \\
= \|F'(x^*)^{-1}([x_k, x^*] - [x_k, x_k] + [x_k, x_k] - [x_k, x_k]) - [x_k, x_k] + [x_k, x_k, x_k]\| \\
\leq \|F'(x^*)^{-1}([x_k, x^*] - [x_k, x_k])\| + \|F'(x^*)^{-1}([x_k, x_k, x_k] - [x_k, x_k, x_k]) - [x_k, x_k, x_k] + [x_k, x_k, x_k])\| \\
\leq b_0\|x_k - x^*\| + c\|x_k - x_k\| \|x_k - x_k\| \\
(75) \quad \leq b_0\|x_k - x^*\| + c\|x_k - x^*\| + \|x^* - x_k\| \|x_k - x_k\| \|x_k - x_k\|.
\]

By (2) we can write

\[
\|x_k + x^*\| = \|x_k - x^* - L_k^{-1}(F(x_k) - F(x^*))\| \\
= \| - L_k^{-1}([x_k, x^*] - L_k)(x_k - x^*)\| \\
(76) \quad \leq \|L_k^{-1}F'(x^*)\| \|F'(x^*)^{-1}([x_k, x^*] - L_k)\| \|x_k - x^*\|.
\]

Estimate (72) now follows from (74), (75) and (76). By the choice of \( r^* \) and (76) we get

\[
\|x_k + x^*\| < \|x_k - x^*\| < r^*,
\]

from which it follows \( x_k + 1 U(x^*, r^*) \) and \( \lim_{k \to \infty} x_k = x^* \).

That completes the proof of Theorem 3. \( \square \)

**Remark 2.** In the elegant paper [12] the following conditions were used:

\[
(77) \quad \|F'(x^*)^{-1}([x, y] - [u, v])\| \leq b\|x - u\| + \|y - v\|
\]

and

\[
(78) \quad \|F'(x^*)^{-1}([u, v, x, y])\| \leq c\|u - v\|
\]

for all \( x, y, u, v \in D \). Note that (77) implies \( F'(x^*) = [x^*, x^*] \), [4], [8].

The convergence radius is given by

\[
r_p = \frac{2}{3b + \sqrt{9b^2 + 16c}}.
\]
Moreover the corresponding error bounds are
\[
\|x_{n+1} - x^*\| \leq \beta_n \\
\leq \frac{6\|x_n - x^*\| + c(\|x_n - x^*\| + \|x_{n-2} - x^*\|)\left(\|x_{n-1} - x^*\| + \|x_n - x^*\|\right)}{1 - 2b\|x_n - x^*\| - c(\|x_n - x^*\| + \|x_{n-2} - x^*\|)}\|x_n - x^*\|, \quad n \geq 0. 
\]
It can easily be seen that conditions (65)–(69) are weaker than (77) and (78). Moreover in general
\[
a_0 \leq b_0 \leq b \quad \text{and} \quad c_0 \leq c. 
\]
Hence,
\[
r_p \leq r^* 
\]
and
\[
\alpha_n \leq \beta_n, \quad n \geq 0. 
\]
In case strict inequality holds in one of (81) or (82) then (83), (84) hold as strict inequalities also. That is under weaker conditions we provide finer error bounds and a wider choice of initial guesses. This observation is important in computational mathematics.

Finally note that it was also shown in [12] that the \(R\)-order of convergence of method (2) is the unique positive root of equation
\[
t^3 - t^2 - t - 1 = 0, 
\]
being approximately 1.839 . . .

REFERENCES


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