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A CONVERGENCE ANALYSIS OF AN ITERATIVE ALGORITHM OF ORDER 1.839... UNDER WEAK ASSUMPTIONS

IOANNIS K. ARGYROS*

Abstract. We provide new and weaker sufficient local and semilocal conditions for the convergence of a certain iterative method of order 1.839... to a solution of an equation in a Banach space. The new idea is to use center-Lipschitz/Lipschitz conditions instead of just Lipschitz conditions on the divided differences of the operator involved. This way we obtain finer error bounds and a better information on the location of the solution under weaker assumptions than before.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet differentiable operator on an open convex subset D of a Banach space X with values in a Banach space Y.

The iterations

$$\begin{aligned} x_{n+1} &= x_n - L_n^{-1} F(x_n), \\ L_n &= [x_n, x_{n-1}] + [x_{n-2}, x_n] - [x_{n-2}, x_{n-1}], \quad n \geq 0, \end{aligned} \tag{2}$$

have been already used to generate a sequence converging to x^* with R-order 1.839... [4], [12] and [13]. Here, $[x, y] \in L(X, Y)$, $[x, y, z] \in L(X, L(X, Y))$ denote divided differences of order one and two respectively of operator F satisfying

$$[x, y](y - x) = F(y) - F(x) \tag{3}$$

and

$$[x, y, z](y - z) = [x, y] - [x, z] \tag{4}$$

for all $x, y, z \in D$ [4].

*Department of Mathematics, Cameron University, Lawton, OK 73505, USA, e-mail: ioannisa@cameron.edu.

Method (2) is considered to be a discretized version of the famous cubically convergent methods of Euler–Chebyshev (tangent hyperbola) or Halley (parabola) [1]–[3], [6], [10] and [11]. Discretized versions of the above methods using divided differences of order one and two or just one have also been considered in [5], [7] and [9].

Here we provide a new local and semilocal convergence analysis for method (2). Using Lipschitz and center Lipschitz conditions on the divided differences of operator F instead of just Lipschitz conditions we introduce weaker sufficient convergence conditions than before. Moreover we obtain finer error bounds on the distances involved as well as a better information on the location of the solution x^*. Furthermore in the case of local analysis a larger convergence radius is obtained.

2. SEMILOCAL ANALYSIS

Let d_i, $i = 0, 1, \ldots, 5$, η_0, η_1 be non-negative parameters and $\delta \in [0, 1)$. Define the parameters α_0, α_1, α_2, β_0, β_1 by

$$\alpha_2 = (1-\delta^2)d_5, \tag{5}$$
$$\alpha_1 = \delta\{\delta(d_1+\delta d_0) - (1-\delta^2)(d_3+\delta d_2)\}, \tag{6}$$
$$\alpha_0 = -\delta(1-\delta)(\eta_0+\eta_1)d_5\eta_1, \tag{7}$$
$$\beta_1 = d_3 + \delta(d_0+d_2), \tag{8}$$
$$\beta_0 = (\eta_0+\eta_1)\eta_1(d_5+\delta d_4) - \delta, \tag{9}$$

and functions f, g by

$$f(t) = \alpha_2 t^2 + \alpha_1 t + \alpha_0, \tag{10}$$
$$g(t) = \beta_1 t + \beta_0. \tag{11}$$

We can show the following result on majorizing sequences.

THEOREM 1. *Let η be a non-negative parameter such that*

$$\eta \le \begin{cases} \min\{\alpha_3, \beta_2\},\ \beta_2 = -\frac{\beta_0}{\beta_1}, & \text{if } \beta_1 \neq 0, \\ 0, & \text{if } \alpha_0 = 0, \\ \alpha_3, & \text{if } \beta_1 = 0, \end{cases} \tag{12}$$

provided

$$\beta_0 \le 0, \tag{13}$$

where α_3, β_2 are the non-negative zeros of functions f and g respectively.

Then

(a) *Iteration* $\{t_n\}$, $n \geq -2$, *given by*

$$\begin{aligned} t_{-2} =& 0, \\ t_{-1} =& \eta_0, \\ t_0 =& \eta_0 + \eta_1, \\ t_1 =& \eta_0 + \eta_1 + \eta, \end{aligned}$$

$$t_{n+2} = t_{n+1} + \frac{d_3(t_{n+1}-t_n)+d_5(t_n-t_{n-2})(t_n-t_{n-1})}{1-d_4\eta_1(\eta_0+\eta_1)-d_0(t_{n+1}-t_0)-d_1(t_n-t_0)-d_2(t_{n+1}-t_n)}(t_{n+1}-t_n), \quad n \geq 0, \tag{14}$$

is non-decreasing, bounded above by

$$t^{**} = \frac{\eta}{1-\delta} + \eta_0 + \eta_1 \tag{15}$$

and converges to t^* *such that*

$$0 \leq t^* \leq t^{**}. \tag{16}$$

Moreover, the following error bounds hold for all $n \geq 0$

$$0 \leq t_{n+2} - t_{n+1} \leq \delta(t_{n+1} - t_n) \leq \delta^{n+1}\eta. \tag{17}$$

(b) *Iteration* $\{s_n\}$, $n \geq -2$, *given by*

$$\begin{aligned} s_{-2} - s_{-1} =& \eta_1, \\ s_{-1} - s_0 =& \eta_0, \\ s_0 - s_1 =& \eta, \end{aligned}$$

$$s_{n+1} - s_{n+2} = \frac{d_3(s_n-s_{n+1})+d_5(s_{n-2}-s_n)(s_{n-1}-s_n)}{1-d_4\eta_1(\eta_0+\eta_1)-d_0(s_0-s_{n+1})-d_1(s_0-s_n)-d_2(s_n-s_{n+1})}(s_n-s_{n+1}), \quad n \geq 0, \tag{18}$$

for s_{-1}, s_0, $s_1 \geq 0$ *is non-increasing, bounded below by*

$$s^{**} = s_0 - \frac{\eta}{1-\delta} \tag{19}$$

and converges to some s^* *such that*

$$0 \leq s^{**} \leq s^*. \tag{20}$$

Moreover, the following error bounds hold for all $n \geq 0$:

$$0 \leq s_{n+1} - s_{n+2} \leq \delta(s_n - s_{n+1}) \leq \delta^{n+1}\eta. \tag{21}$$

Proof. (a) We must show:

$$(d_3 + \delta d_2)(t_{k+1} - t_k) + d_5(t_k - t_{k-2})(t_k - t_{k-1}) + \delta d_0(t_{k+1} - t_0) + \delta d_1(t_k - t_0) + \delta d_4\eta_1(\eta_0 + \eta_1) \leq \delta \tag{22}$$

$$1 - d_4\eta_1(\eta_0 + \eta_1) - d_0 t_{k+1} - d_1 t_k - d_2(t_{k+1} - t_k) > 0 \tag{23}$$

for all $k \geq 0$.

Inequalities (22) and (23) hold for $k=0$ by the initial conditions. But then (14) gives

$$0 \le t_2 - t_1 \le \delta(t_1 - t_0). \tag{24}$$

Let us assume (22), (23) and (17) hold for all $k \le n+1$. By the induction hypotheses we get

$$\begin{aligned} &(d_3 + \delta d_2)(t_{k+2} - t_{k+1}) + d_5(t_{k+1} - t_{k-1})(t_{k+1} - t_k) + \delta d_0(t_{k+2} - t_0) \\ &\quad + \delta d_1(t_{k+1} - t_0) + \delta d_4 \eta_1(\eta_0 + \eta_1) \le \\ &\le (d_3 + \delta d_2)\delta^{k+1}\eta + d_5(\delta^k + \delta^{k-1})\eta^2\delta^k + \delta d_0 \tfrac{\eta}{1-\delta} + \delta d_1 \tfrac{\eta}{1-\delta} + \delta d_4 \eta_1(\eta_0 + \eta_1). \end{aligned} \tag{25}$$

It is clear that (25) will be bounded above by δ if

$$\begin{aligned} &(d_3 + \delta d_2)\delta^{k+1}\eta + d_5(\delta^k + \delta^{k-1})\eta^2\delta^k + \delta d_0 \tfrac{\eta}{1-\delta} + \delta d_1 \tfrac{\eta}{1-\delta} + \delta d_4 \eta_1(\eta_0 + \eta_1) \le \\ &\le (d_3 + \delta d_2)\eta + d_5(\eta_0 + \eta_1)\eta_1 + \delta d_0 \eta + \delta d_4 \eta_1(\eta_0 + \eta_1) \end{aligned}$$

or

$$\begin{aligned} &(d_3 + \delta d_2)(1-\delta)\delta^{k+1}\eta + d_5\delta^{k-1}(1-\delta^2)\eta^2\delta^k + \delta d_0 \eta + \delta d_1 \eta \le \\ &\le (1-\delta)(d_3 + \delta d_2)\eta + (1-\delta)d_5(\eta_0 + \eta_1)\eta_1 + (1-\delta)\delta d_0 \eta \end{aligned}$$

or, for $k \ge 0$,

$$\begin{aligned} &(d_3 + \delta d_2)\delta(1-\delta)\eta + \tfrac{1-\delta^2}{\delta} d_5 \eta^2 + \delta d_1 \eta + \delta d_0 \eta \le \\ &\le (d_3 + \delta d_2)(1-\delta)\eta + d_5(1-\delta)(\eta_0 + \eta_1)\eta_1 + (1-\delta)\delta d_0 \eta \end{aligned}$$

or, for $\delta \ne 0$,

$$\alpha_2 \eta^2 + \alpha_1 \eta + \alpha_0 \le 0, \tag{26}$$

which is true by the choice of η. By the same proof as above we show (23) for $k = n+1$.

We must also show:

$$t_k \le t^{**}, \quad k \ge 1. \tag{27}$$

For $k = 1, 2$ we have $t_1 \le t^*$ and $t_2 \le t_1 + \delta\eta = \eta_0 + \eta_1 + (1+\delta)\eta \le t^{**}$. Assume (27) holds for all $k \le n+1$. It follows from (17) that

$$\begin{aligned} t_{k+2} &\le t_{k+1} + \delta(t_{k+1} - t_k) \\ &\le t_k + \delta(t_k - t_{k-1}) + \delta(t_{k+1} - t_k) \\ &\quad \dots \\ &\le t_1 + \delta(t_1 - t_0) + \dots + \delta(t_{k+1} - t_k) \\ &\le \eta_0 + \eta_1 + \eta + \delta\eta + \dots + \delta^{k+1}\eta \\ &= \eta_0 + \eta_1 + \tfrac{1-\delta^{k+2}}{1-\delta}\eta \\ &< t^{**}. \end{aligned}$$

That is, $\{t_n\}$, $n \ge 0$, is bounded above t^{**}. By (22) and (23) we get

$$t_{k+2} - t_{k+1} \ge 0. \tag{28}$$

Hence sequence $\{t_n\}$, $n \geq 0$, is also non-decreasing and as such it converges to some t^* satisfying (16).

(b) The proof follows along the lines of part (a).

That completes the proof of Theorem 1. □

We show the following semilocal convergence theorem for method (2).

THEOREM 2. *Let $F\colon D \subseteq X \to Y$ be a differentiable operator with divided differences of the first and second order denoted by $[\cdot,\cdot]$, $[\cdot,\cdot,\cdot]$ respectively. Assume:*

– *there exist points $x_{-2}, x_{-1}, x_0 \in D$ so L_0 is invertible and non-negative numbers η, d_i, $i = 0, 1, \dots, 5$ such that*

$$\|L_0^{-1}([x_0,x_0]-[y,x_0])\| \leq d_0\|x_0-y\|, \tag{29}$$

$$\|L_0^{-1}([x,x_0]-[x,y])\| \leq d_1\|x_0-y\|, \tag{30}$$

$$\|L_0^{-1}([x,y]-[x,z])\| \leq d_2\|y-z\|, \tag{31}$$

$$\|L_0^{-1}([x,y]-[x,x])\| \leq d_3\|x-y\|, \tag{32}$$

$$\|L_0^{-1}([x,y,x_0]-[z,y,x_0])\| \leq d_4\|x-z\|, \tag{33}$$

$$\|L_0^{-1}([x,x,y]-[z,x,y])\| \leq d_5\|x-z\|, \quad \textit{for all } x,y,z \in D, \tag{34}$$

$$\|x_{-1}-x_0\| \leq \eta_1, \quad \|x_{-1}-x_{-2}\| \leq \eta_0, \quad \|x_1-x_0\| \leq \eta; \tag{35}$$

– *hypotheses of Theorem 1 hold, and*

$$\overline{U}(x_0,t^*) = \{x \in X : \|x-x_0\| \leq t^*\} \subseteq D. \tag{36}$$

Then method $\{x_n\}$, $n \geq 0$, generated by (2) is well defined, remains in $\overline{U}(x_0,t^*)$ for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}(x_0,t^*)$ of equation $F(x) = 0$. Moreover, the following error bounds hold for all $n \geq 0$:

$$\|x_{n+2}-x_{n+1}\| \leq t_{n+2}-t_{n+1} \tag{37}$$

and

$$\|x_n - x^*\| \leq t^* - t_n. \tag{38}$$

Furthermore, if there exists $R \geq t^*$ such that

$$U(x_0,R) \subseteq D, \tag{39}$$

$$\|L_0^{-1}([x,y]-[z,w])\| \leq d_6(\|x-z\|+\|y-w\|), \quad \textit{for all } x,y,z,w \in D, \tag{40}$$

and

$$d_6(R+t^*+2\eta_0+2\eta_1) \leq 1 \tag{41}$$

or $[\cdot,\cdot]$ is symmetric and

$$d_6(R+t^*+\eta_0+\eta_1)+d_1(\eta_0+\eta_1) \leq 1, \tag{42}$$

the solution x^* is unique in $U(x_0,R)$.

Proof. Let us prove

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \quad k \geq -2. \tag{43}$$

For $k = -2, -1, 0$ (43) holds by the initial conditions. Assume (43) holds for all $n \leq k$. Using (29), (30), (31), (33) and (43) we obtain

$$\begin{aligned}
& \|L_0^{-1}(L_0 - L_{k+1})\| = \\
= \; & \|L_0^{-1}(L_0 - [x_0, x_0] + [x_0, x_0] - [x_{k+1}, x_0] + [x_{k+1}, x_0] \\
& - [x_{k+1}, x_k] + [x_{k+1}, x_k] - L_{k+1})\| \\
\leq \; & \|L_0^{-1}([x_0, x_{-1}, x_0] - [x_{-2}, x_{-1}, x_0])(x_{-1} - x_0)\| \\
& + \|L_0^{-1}([x_0, x_0] - [x_{k+1}, x_0])\| + \|L_0^{-1}([x_{k+1}, x_0] - [x_{k+1}, x_k])\| \\
& + \|L_0^{-1}([x_{k-1}, x_k] - [x_{k-1}, x_{k+1}])\| \\
\leq \; & d_4\|x_{-1} - x_0\| \, \|x_0 - x_{-2}\| + d_0\|x_{k+1} - x_0\| \\
& + d_1\|x_k - x_0\| + d_2\|x_k - x_{k+1}\| \\
\leq \; & d_4\eta_1(\eta_0 + \eta_1) + d_0(t_{k+1} - t_0) + d(t_k - t_0) + d_2(t_{k+1} - t_k) \\
\leq \; & d_4\eta_1(\eta_0 + \eta_1) + d_0(t^* - t_0) + d_1(t^* - t_0) + d_2\delta^k\eta \\
< \; & 1. \qquad (44)
\end{aligned}$$

by the choice of δ and (12).

It follows by (44) and the Banach Lemma on invertible operators [8] that L_{k+1}^{-1} is invertible and

$$\begin{aligned}
& \|L_{k+1}^{-1}L_0\| \leq \\
& \leq \big[1 - (d_4\|x_{-1} - x_0\| \, \|x_0 - x_{-2}\| + d_0\|x_{k+1} - x_0\| + d_1\|x_k - x_0\| \\
& \quad + \; d_2\|x_{k+1} - x_k\|)\big]^{-1} \\
& \leq [1 - d_4\eta_1(\eta_0 + \eta_1) - d_0(t_{k+1} - t_0) - d_1(t_k - t_0) - d_2(t_{k+1} - t_k)]^{-1}. \qquad (45)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
& \|L_0^{-1}([x_k, x_{k+1}] - L_k)\| \\
& = \|L_0^{-1}([x_k, x_{k+1}] - [x_k, x_k] + [x_k, x_k] - L_k)\| \\
& \leq \|L_0^{-1}([x_k, x_{k+1}] - [x_k, x_k])\| \\
& \quad + \|L_0^{-1}([x_k, x_k, x_{k-1}] - [x_{k-2}, x_k, x_{k-1}])(x_k - x_{k-1})\| \\
& \leq d_3\|x_{k+1} - x_k\| + d_5\|x_k - x_{k-2}\| \, \|x_k - x_{k-1}\| \\
& \leq d_3(t_{k+1} - t_k) + d_5(t_k - t_{k-2})(t_k - t_{k-1}). \qquad (46)
\end{aligned}$$

Furthermore using (2), (43), and (46) we get

$$\|x_{k+2} - x_{k+1}\| \leq \|L_{k+1}^{-1}L_0\| \, \|L_0^{-1}([x_k, x_{k+1}] - L_k)\| \, \|x_{k+1} - x_k\| \leq t_{k+2} - t_{k+1}, \tag{47}$$

which shows (43).

Theorem 1 and (47) imply $\{x_n\}$, $n \geq 0$, is a Cauchy sequence in a Banach space X and as such it converges to some $x^* \in \overline{U}(x_0, t^*)$, since $\overline{U}(x_0, t^*)$ is a closed set.

By (43) we get

$$\|x_n - x_m\| \leq t_m - t_n, \quad -2 \leq n \leq m, \tag{48}$$

while by letting $m \to \infty$ in (48) we obtain (38).

Finally, by letting $k \to \infty$ in (47), we get $F(x^*) = 0$. To show uniqueness, let $y^* \in U(x_0, R)$ be a solution of equation $F(x) = 0$. We can have in turn

$$\begin{aligned}
&\|L_0^{-1}([y^*, x^*] - L_0)\| \leq \\
&\leq \|L_0^{-1}([y^*, x^*] - [x_0, x_{-2}])\| \\
&\quad + \|L_0^{-1}([x_{-2}, x_{-1}] - [x_{-1}, x_0])\| \\
&\leq d_6(\|y^* - x_0\| + \|x^* - x_{-2}\| + \|x_{-2} - x_{-1}\| + \|x_{-1} - x_0\|) \\
&< d_6(R + t^* + 2\eta_0 + 2\eta_1) \\
&\leq 1. && (49)
\end{aligned}$$

It follows by (49) and the Banach Lemma on invertible operators that $[y^*, x^*]$ is invertible. Hence, from

$$F(x^*) - F(y^*) = [y^*, x^*](x^* - y^*), \tag{50}$$

we deduce that $x^* = y^*$.

If $[\cdot, \cdot]$ is symmetric as in (49) we get

$$\|L_0^{-1}([y^*, x^*] - L_0)\| < d_6(R + t^* + \eta_0 + \eta_1) + d_1(\eta_0 + \eta_1) \leq 1. \tag{51}$$

We conclude again that $x^* = y^*$.

That completes the proof of Theorem 2. □

The proof of the following result follows exactly as in Theorem 2 but using part (b) of Theorem 1.

THEOREM 3. *Assume hypotheses of Theorems* 1 *and* 2 *hold.*

Then method $\{x_n\}$, $n \geq 0$, *generated by* (2) *is well defined, remains in* $\overline{U}(x_0, s^*)$ *for all* $n \geq 0$ *and converges to a solution* $x^* \in \overline{U}(x_0, s^*)$ *of equation* $F(x) = 0$. *Moreover, the following error bounds hold for all* $n \geq 0$:

$$\|x_{n+2} - x_{n+1}\| \leq s_{n+1} - s_{n+2} \tag{52}$$

and

$$\|x_n - x^*\| \leq s_n - s^*. \tag{53}$$

Furthermore if there exists $R_1 \geq s^*$ *such that, together with* (40),

$$U(x_0, R_1) \subseteq D \quad \textit{holds}, \tag{54}$$

and

$$d_6[R_1 + s^* + 2(\eta_0 + R_1)] \leq 1 \tag{55}$$

or

$$[\cdot,\cdot] \quad \textit{is symmetric}$$

and

$$d_6(R_1 + s^* + \eta_0 + \eta_1) + d_1(\eta_0 + \eta_1) \le 1, \tag{56}$$

the solution x^* is unique in $U(x_0, R)$.

REMARK 1. In [12, Th. 5.1], condition (40) was used together with

$$\|L_0^{-1}([x,y,z] - [u,y,z])\| \le d_7\|x - u\| \tag{57}$$

for all $x, y, z, v \in D$ to show convergence of method (2).

The following error bounds were found

$$\|x_{n+1} - x_n\| \le v_n - v_{n+1} \tag{58}$$

and

$$\|x_n - x^*\| \le v_n - v^*, \tag{59}$$

where,

$$v^* = \lim_{n\to\infty} v_n, \tag{60}$$

and $\{v_n\}$ is similar to $\{s_n\}$ but using d_6, d_7, η_0, η_1, η instead of d_0, d_1, d_2, d_3, d_4, d_5, η_0, η_1, η. Note also that in general

$$d_0 \le d_1 \le d_3 \le d_2 \le d_6 \tag{61}$$

and

$$d_4 \le d_5 \le d_7. \tag{62}$$

Hence we can easily obtain by induction

$$s_n - s_{n+1} \le v_n - v_{n+1} \tag{63}$$

and

$$s_n - s^* \le v_n - v^*. \tag{64}$$

That is, under weaker convergence conditions we obtain finer error bounds.□

3. LOCAL ANALYSIS

We can show the following local results for method (2).

Theorem 4. *Let $F\colon D\subseteq X\to Y$ be a differentiable operator. Assume F has divided differences of the first and second order such that:*

$$F'(x^*) = [x^*,x^*], \tag{65}$$
$$\|F'(x^*)^{-1}([x^*,x^*]-[x,x^*])\| \le a_0\|x-x^*\|, \tag{66}$$
$$\|F'(x^*)^{-1}([x,x^*]-[x,y])\| \le b_0\|x-x^*\|, \tag{67}$$
$$\|F'(x^*)^{-1}([x,x^*,y]-[z,x^*,y])\| \le c_0\|x-z\|, \tag{68}$$
$$\|F'(x_0)^{-1}([u,x,y]-[v,x,y])\| \le c\|u-v\|, \tag{69}$$
$$\overline{U}(x^*,r^*) \subseteq D, \tag{70}$$

for all $x,y,z,u,v\in D$, where x^* is a simple zero of F, and

$$r^*=\frac{2}{a_0+2b_0+\sqrt{(a_0+2b_0)^2+8(c+c_0)}}\,. \tag{71}$$

Then method (2) is well defined, remains in $U(x^*,r^*)$ for all $n\ge 0$, and converges to x^* provided that x_{-1}, x_{-2}, $x_0\in U(x^*,r^*)$.

Moreover, the following error bounds hold for all $n\ge 0$:

$$\|x_{n+1}-x^*\| \le$$
$$\le \frac{b_0\|x_n-x^*\|+c\big(\|x_n-x^*\|+\|x_{n-2}-x^*\|\big)\big(\|x_{n-1}-x^*\|+\|x_n-x^*\|\big)}{1-(a_0+b_0)\|x_n-x^*\|-c_0\big(\|x_n-x^*\|+\|x_{n-2}-x^*\|\big)\|x_{n-1}-x^*\|}\|x_n-x^*\|=\alpha_n. \tag{72}$$

Proof. We first show linear operator

$$L\equiv L(x,y,z)=[x,y]+[z,x]-[z,y],\quad x,y,z\in U(x^*,r^*)$$

is invertible. By (65), (67), (69), we get in turn

$$\begin{aligned}
&\|F'(x^*)^{-1}(F'(x^*)-L)\| =\\
&=\|F'(x^*)^{-1}\{[x^*,x^*]-[x,x^*]+[z,x^*]-[z,x]\\
&\qquad\qquad +[x,x^*]-[x,y]-[z,x^*]+[z,y]\}\|\\
&\le \|F'(x^*)^{-1}([x^*,x^*]-[x,x^*])\|+\|F'(x^*)^{-1}([z,x^*]-[z,x])\|\\
&\quad +\|F'(x^*)^{-1}([x,x^*,y]-[z,x^*,y])(x^*-y)\|\\
&\le (a_0+b_0)\|x-x^*\|+c\|x-z\|\cdot\|x^*-y\|\\
&\le (a_0+b_0)\|x-x^*\|+c(\|x-x^*\|+\|z-x^*\|)\|y-x^*\|\\
&< (a_0+b_0)r^*+2c(r^*)^2\\
&< 1,
\end{aligned} \tag{73}$$

by the choice of r^*. It follows from (73) and the Banach Lemma on invertible operators that L is invertible.

Assume $x_{k-2},\ x_{k-1},\ x_k \in U(x^*, r^*)$ and set $x_k = x$, $x_{k-1} = y$, $x_{k-2} = z$, $k = 0, 1, 2, \dots, n$, $L_k = L(x_k, x_{k-1}, x_{k-2})$. We have

$$\|L_k^{-1}F'(x^*)\| \le \big[1 - (a_0 + b_0)\|x_k - x^*\| - c_0\big(\|x_k - x^*\| + \|x_{k-2} - x^*\|\big)\|x_{k-1} - x^*\|\big]^{-1}. \tag{74}$$

Moreover, by (67) and (69) we get

$$\begin{aligned}
&\|F'(x^*)^{-1}([x_k, x^*] - L_k)\| = \\
&= \|F'(x^*)^{-1}\big([x_k, x^*] - [x_k, x_k] + [x_k, x_k] - [x_k, x_{k-1}] \\
&\quad - [x_{k-2}, x_k] + [x_{k-2}, x_{k-1}]\big)\| \\
&\le \|F'(x^*)^{-1}\big([x_k, x^*] - [x_k, x_k]\big)\| \\
&\quad + \|F'(x^*)^{-1}\big([x_k, x_k, x_{k-1}] - [x_{k-2}, x_k, x_{k-1}]\big)(x_k - x_{k-1})\| \\
&\le b_0\|x_k - x^*\| + c\|x_k - x_{k-2}\|\,\|x_k - x_{k-1}\| \\
&\le b_0\|x_k - x^*\| + c\big(\|x_k - x^*\| + \|x^* - x_{k-2}\|\big)\big(\|x_k - x^*\| + \|x^* - x_{k-1}\|\big).
\end{aligned} \tag{75}$$

By (2) we can write

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|x_k - x^* - L_k^{-1}(F(x_k) - F(x^*))\| \\
&= \| - L_k^{-1}([x_k, x^*] - L_k)(x_k - x^*)\| \\
&\le \|L_k^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}([x_k, x^*] - L_k)\|\,\|x_k - x^*\|.
\end{aligned} \tag{76}$$

Estimate (72) now follows from (74), (75) and (76). By the choice of r^* and (76) we get

$$\|x_{k+1} - x^*\| < \|x_k - x^*\| < r^*,$$

from which it follows $x_{k+1} \in U(x^*, r^*)$ and $\lim_{k\to\infty} x_k = x^*$.

That completes the proof of Theorem 3. □

Remark 2. In the elegant paper [12] the following conditions were used:

$$\|F'(x^*)^{-1}\big([x, y] - [u, v]\big)\| \le b\big(\|x - u\| + \|y - v\|\big) \tag{77}$$

and

$$\|F'(x^*)^{-1}\big([u, x, y] - [v, x, y]\big)\| \le c\|u - v\| \tag{78}$$

for all $x, y, u, v \in D$. Note that (77) implies $F'(x^*) = [x^*, x^*]$, [4], [8].

The convergence radius is given by

$$r_p = \frac{2}{3b + \sqrt{9b^2 + 16c}}\,. \tag{79}$$

Moreover the corresponding error bounds are

$$\begin{aligned}(80)\quad &\|x_{n+1}-x^*\| \leq \\ &\leq \beta_n \\ &= \frac{b\|x_n-x^*\|+c\big(\|x_n-x^*\|+\|x_{n-2}-x^*\|\big)\big(\|x_{n-1}-x^*\|+\|x_n-x^*\|\big)}{1-2b\|x_n-x^*\|-c\big(\|x_n-x^*\|+\|x_{n-2}-x^*\|\big)\|x_{n-1}-x^*\|}\|x_n-x^*\|, \quad n\geq 0.\end{aligned}$$

It can easily be seen that conditions (65)–(69) are weaker than (77) and (78). Moreover in general

$$a_0 \leq b_0 \leq b \tag{81}$$

and

$$c_0 \leq c. \tag{82}$$

Hence,

$$r_p \leq r^* \tag{83}$$

and

$$\alpha_n \leq \beta_n, \quad n \geq 0. \tag{84}$$

In case strict inequality holds in one of (81) or (82) then (83), (84) hold as strict inequalities also. That is under weaker conditions we provide finer error bounds and a wider choice of initial guesses. This observation is important in computational mathematics.

Finally note that it was also shown in [12] that the R-order of convergence of method (2) is the unique positive root of equation

$$t^3 - t^2 - t - 1 = 0,$$

being approximately $1.839\ldots$.

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