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SOLVING DISCRETE MULTIATTRIBUTE PROBLEMS UNDER RISK BASED ON AN APPROXIMATION*

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Abstract. We consider the multiattribute decision making problem under risk with imprecise information on the decision maker's preferences, modelled by means of a vector utility function. We propose an interactive decision aid approach, which uses an idea of approximation to the utility efficient set and qualitative comparisons for the decision maker, to overcome the possible difficulty in generate it. An application to university selection illustrates the procedure.

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1. INTRODUCTION

We consider the multiattribute decision making problem under risk with partial information on the DM's (decision maker) preferences, in the sense that his/her preferences are modelled by means of a vector utility function, Roberts (1972, 1979) and Rietveld (1980) instead of a scalar one, Fishburn (1976) and Keeney and Raiffa (1993). This vector utility function represents imprecise preferences and can be seen as a way for lack of precision of the true but unknown scalar utility function, which is usually defined on the attributes associated with the lowest-level objectives of a hierarchy, which often exhibit well defined multi-objective problems. In other words, if no independence assumption is accepted for the DM to reach a more structured form in the utility function (additive, multiplicative, ...), he/she has to cope with such a vector function in the decision making process.

In this framework, the utility efficient set, Ríos-Insua and Mateos (1997), plays a fundamental role because of its property: from any strategy in this set, it is not possible to feasibly move in order to increase one component without necessarily decreasing at least one of the remainder. However, the generation of such set can be involved, specially in continuous problems. In the case of discrete problems it is easier, but we note that the set of utility efficient solutions might be far too big in most practical situations. Hence, the

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generation of the utility efficient set is not usually considered the resolution of the multiattribute problem because this set may have many elements and is not generally totally ordered. Thus, there is a need for more intelligent strategies to generate a set of representative efficient solutions or an approximation that gives a fair representation of the whole set. Although there are several methods to aid DMs to generate the efficient set or a representative subset, see, e.g., Gal (1972), Goicoechea et al. (1982), Yu (1985), Chankong and Haimes (1983), Steuer (1986), Vincke (1992), Gal (1995) and Gal et al. (1999), there is not a definitive solution to this problem, particularly for problems under risk.

In this paper we consider a utility-based procedure, see e.g., von Neumann and Morgenstern (1947) or Keeney and Raiffa (1993), which uses an approximation concept, Mateos and Ríos-Insua (1996), which intends, on one hand, to facilitate the generation process of a representative set of the whole utility efficient set and, on the other, its interactive reduction to reach a final strategy.

Throughout the paper we shall employ the following notation: For two scalars a and b, $a \geqq b$ denotes $a > b$ or $a = b$. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \geqq \mathbf{y}$ denotes $x_i \geqq y_i$ for $i = 1, \dots, n$, and $\mathbf{x} \geq \mathbf{y}$ denotes $\mathbf{x} \geqq \mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$. Our framework is the multiattribute decision making problem under risk with a finite set Z of outcomes $\mathbf{z}^i$, $i = 1, \dots, s$, characterized by a number of attributes $z_1, \dots, z_N$ and a set $\mathcal{P}_Z$ of simple probability distributions over Z, with elements $p, q, \dots$, called strategies or (risky) prospects, where for example, $p = (p_1, \mathbf{z}^1; p_2, \mathbf{z}^2; \dots; p_s, \mathbf{z}^s)$, $p_j \geqq 0$ for all j and $\sum_{j=1}^{s} p_j = 1$. We assume partial information in the sense that we can assess a vector utility function $\mathbf{u} : Z \to \mathbb{R}^m$, where $\mathbf{u} = (u_1, \dots, u_m)$ represents a preference order $\succ$ (asymmetric and transitive) on $\mathcal{P}_Z$, leading to a dominance principle defined by

$$p \succ q \;\Leftrightarrow\; E\left(\mathbf{u}, p\right) \geq E\left(\mathbf{u}, q\right),$$

where $E(\mathbf{u}, p) = \big(E\left(u_1, p\right), \dots, E\left(u_m, p\right)\big)$ is the expected utility vector with

$$E\left(u_i, p\right) = \sum_{j=1}^{s} p_j u_i(\mathbf{z}^j), \quad \text{for } i = 1, \dots, m.$$

Throughout the paper, we will assume that the real-valued functions u_i, $i = 1, \dots, m$, on Z are continuous, monotonous and bounded. This framework leads to the vector maximum problem over $\mathcal{P}_Z$, defined

$$\max\left\{E\left(\mathbf{u}, p\right) : p \in \mathcal{P}_Z\right\}. \tag{1}$$

The natural solution concept is the utility efficiency: $p \in \mathcal{P}_Z$ is a *utility efficient strategy* if there is no $q \in \mathcal{P}_Z$ such that $E\left(\mathbf{u}, q\right) \geq E\left(\mathbf{u}, p\right)$. Such set of strategies will be called the *utility efficient set* and denoted by $\mathcal{E}(\mathcal{P}_Z, \mathbf{u})$. Note that this definition extends the one for problems under certainty, i.e., if $\mathcal{P}_Z$ consists of sure prospects $p_{\mathbf{z}} = (1, \mathbf{z})$, then $\mathcal{E}(\mathcal{P}_Z, \mathbf{u}) = \mathcal{E}(Z, \mathbf{u})$, where $\mathcal{E}(Z, \mathbf{u})$ would be the efficient set for Z given $\mathbf{u}$.

It is clear that if $\mathcal{E}(\mathcal{P}_Z, \mathbf{u})$ has a unique element p, it would be the compromise strategy for the DM. However, as mentioned above, this is not the case for most real problems, as $\mathcal{E}(\mathcal{P}_Z, \mathbf{u})$ may have a lot of strategies and its generation can be difficult. Thus, we shall consider an approximation to the utility efficient set that will be easier to generate and that could be interactively reduced in the process of reaching a final strategy.

The paper includes five more sections. In Section 2 we introduce some theory and concepts related to the approximation set. In Section 3, we propose a procedure to reduce the utility efficient set for the case of two utility components, considering a linear preference structure. In Section 4, we extend the procedure to any number of utility components. In Section 5, we present an application of the procedures and, finally, in Section 6, some conclusions are provided.

2. APPROXIMATION OF THE UTILITY EFFICIENT SET

We consider the decision making problem where the DM's preferences are modelled by means of a vector utility function $\mathbf{u} : Z \to \mathbb{R}^m$ and the DM can reveal more information on his preferences through an interactive process, obtaining a more precise vector utility function, as we next shall see.

Let $\mathbf{k}^1 = (k_1^1, \ldots, k_m^1), \ldots, \mathbf{k}^r = (k_1^r, \ldots, k_m^r)$ be the extreme points of the polyhedral of possible weights or scaling constants, which we shall denote by the matrix $M = (\mathbf{k}^1, \ldots, \mathbf{k}^r)^t$. Let us define *the information set* I_M associated to M as

$$I_M = \Big\{\mathbf{k} \in \mathbb{R}^m : \ \mathbf{k} \sum_{i=1}^{r} \alpha_i \mathbf{k}^i, \ \text{with } \alpha = (\alpha_1, \ldots, \alpha_r) \in S_r\Big\},$$

with S_r the simplex on $\mathbb{R}^r$. Now, assume that DM's preferences satisfy the conditions to agree with a (scalar) utility function as a linear function of the components in $\mathbf{u}$, see e.g., French (1986) and Stewart (1996). We define $\mathbf{u}^M = M\mathbf{u}$ =$(\mathbf{k}^1\mathbf{u}, \ldots, \mathbf{k}^r\mathbf{u})$ (each component denotes the inner product of two vectors, and we understand that the first one is a row vector and the second one a column one) as the *vector utility function associated to the information set* I_M. Thus, I_M represents the DM's information on the weights in $\mathbf{u}^M$. Hence, $\mathbf{u}^M$ is an imprecise vector utility function. When $M = M_0$ (the identity matrix of order r) there is *null information* about the scaling constants and we have the original vector utility function $\mathbf{u} = \mathbf{u}^{M_0}$. On the other hand, if $\mathbf{k}^1 = \ldots = \mathbf{k}^r = \mathbf{k}$, we have the scalar utility function $\mathbf{u}^M = \mathbf{k}\mathbf{u}$, with *complete information* about the weights.

From the vector function $\mathbf{u}^M$, we can also consider the corresponding utility efficient set, denoted $\mathcal{E}(\mathcal{P}_Z, \mathbf{u}^M)$. We have the following

Proposition 1. *Given a vector utility function* $\mathbf{u} : Z \to \mathbb{R}^m$ *and an information set* I_M *with* $M = (\mathbf{k}^1, \ldots, \mathbf{k}^r)^t$, *then for all* $p \in \mathcal{P}_Z$

$$E(\mathbf{u}^M, p) = \Big(\mathbf{k}^1 E(\mathbf{u}, p), \ldots, \mathbf{k}^r E(\mathbf{u}, p)\Big) . \tag{2}$$

The proof follows immediately.

Now we shall provide a practical way *to find* $E(\mathbf{u},p^j)$ by means of an algorithm. Let $(i_1,\ldots,i_m)$ be a permutation of the set of indexes $(1,\ldots,m)$ of components in $\mathbf{u}$. We follow the next algorithm to solve the problem $P_{i_1\ldots i_m}$:

Step 1. Let $Y_1=\mathcal{P}_Z$.

Step 2. From $j=1$ to m, solve

$$u^*_{i_j}=\max_{p\in Y_j}E(u_{i_j},p) \tag{3}$$
$$Y_{j+1}=\{p\in Y_j : E(u_{i_j},p)=u^*_{i_j}\}$$

Step 3. End.

Then we obtain the set Y_{m+1}, which usually has a single strategy. Thus, *to find* $E(\mathbf{u},p^j)$ means to solve problem $P_{j,j+1,\ldots,m,1,\ldots,j-1}$, which provides a solution p^j. In analogous way, we can *find* $\mathbf{u}(z^j)$.

Now, we provide the approximation to the utility efficient set for the case of a function with two components $\mathbf{u}=(u_1,u_2)$. Given $\mathbf{u}$ defined over Z, the *approximation set* to $\mathcal{E}(\mathcal{P}_Z,\mathbf{u})$ is defined as

$$\mathcal{A}(\mathcal{P}_Z,\mathbf{u})=\left\{p\in\mathcal{P}_Z : E\left(u_1,p\right)\geqq E\left(u_1,p^2\right) \text{ and } E\left(u_2,p\right)\geqq E\left(u_2,p^1\right)\right\}, \tag{4}$$

where the strategies p^1, p^2 are obtained by solving P_{12} and P_{21}, respectively. As a particular case, we have the definition under certainty, Mateos and Ríos-Insua (1997a, 1997b). Some desirable properties, that fulfils the approximation set (4), are:

1) Let $\mathbf{u}:Z\to\mathbb{R}^2$ be a vector utility function and I_M an information set; then

$$\mathcal{E}(\mathcal{P}_Z,\mathbf{u}^M)\subseteq\mathcal{A}(\mathcal{P}_Z,\mathbf{u}^M)\,. \tag{5}$$

2) *Monotonicity.* Given a vector utility function $\mathbf{u}:Z\to\mathbb{R}^2$ and two information sets, I_{M_1} and I_{M_2}, such that $I_{M_2}\subseteq\operatorname{int}\left(I_{M_1}\right)$, then

$$\mathcal{A}(\mathcal{P}_Z,\mathbf{u}^{M_2})\subseteq\mathcal{A}(\mathcal{P}_Z,\mathbf{u}^{M_1})\,. \tag{6}$$

3) *Convergence.* Let $\mathbf{u}:Z\to\mathbb{R}^2$ be a vector utility function and $\{I_{M_n}\}$ a decreasing sequence of information sets such that $I_{M_1}\subseteq\operatorname{int}(I_{M_0})$ and $\{I_{M_n}\}\downarrow\{\mathbf{q}\}$, when $n\to\infty$. If $p^0\in\mathcal{A}(\mathcal{P}_Z,\mathbf{u}^{M_n})$, for every n, then $p^0\in\mathcal{E}(\mathcal{P}_Z,\mathbf{u})$.

The above definition and properties cannot be extended in immediate way to m-dimensional utilities, i.e. $m\geqq 3$, as we show in next example. Consider the set of outcomes $Z=\{\mathbf{z}^1,\mathbf{z}^2,\mathbf{z}^3,\mathbf{z}^4\}$ and the vector utility function $\mathbf{u}:Z\to\mathbb{R}^3$, such that $\mathbf{u}(\mathbf{z}^1)=(1,2,3)$, $\mathbf{u}(\mathbf{z}^2)=(3,2,1)$, $\mathbf{u}(\mathbf{z}^3)=(1,3,1)$ and $\mathbf{u}(\mathbf{z}^4)=(3,0,3)$. Assume that $\mathcal{P}_Z$ has four strategies $p^1=\left(\frac{2}{3},\mathbf{z}^1;\frac{1}{3},\mathbf{z}^3\right)$,

$p^2 = \left(\frac{2}{3}, \mathbf{z}^2; \frac{1}{3}, \mathbf{z}^3\right)$, $p^3 = \left(\frac{1}{3}, \mathbf{z}^1; \frac{1}{3}, \mathbf{z}^2; \frac{1}{3}, \mathbf{z}^3\right)$ and $p^4 = \left(\frac{1}{2}, \mathbf{z}^3; \frac{1}{2}, \mathbf{z}^4\right)$. The approximation set is $\mathcal{A}(\mathcal{P}_Z, \mathbf{u}) = \{p^1, p^2, p^3\}$, but note that it does not contain to $\mathcal{E}(\mathcal{P}_Z, \mathbf{u}) = \{p^1, p^2, p^3, p^4\}$, which is a fundamental property.

To extend (4) to m-dimensional utilities, let us first consider the two-dimensional vector utility $\mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot))$ and the *nondominated set* by p^i, defined as

$$\mathcal{N}(p^i) = \Big\{p \in \mathcal{P}_Z : \ E(u_j, p) > E(u_j, p^i), \quad j \neq i, \ j = 1, 2\Big\}$$
$$\bigcup \Big\{p \in \mathcal{P}_Z : \ E(u_j, p) = E(u_j, p^i), \quad \forall j\Big\}$$

for $i = 1, 2$. Thus, the approximation set (4) can be rewritten as follows:

$$\mathcal{A}(\mathcal{P}_Z, \mathbf{u}) = \mathcal{N}(p^1) \cap \mathcal{N}(p^2). \tag{7}$$

In analogous way, we can stretch out this idea to the general case of a vector utility function $\mathbf{u}(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$, with $m \geqq 3$, considering the nondominated set

$$\mathcal{N}(p^i) = \Big\{p \in \mathcal{P}_Z : \ \exists j = 1, \ldots, m, \ j \neq i, \text{ with } E(u_j, p) > E(u_j, p^i)\Big\}$$
$$\bigcup \Big\{p \in \mathcal{P}_Z : \ E(u_j, p) = E(u_j, p^i), \ \forall j = 1, \ldots, m\Big\}$$

for each $i = 1, \ldots, m$, being p^i the solution to problem $P_{i,i+1,\ldots,m,1,\ldots,i-1}$. Thus, we define the *approximation set* to the utility efficient set $\mathcal{E}(\mathcal{P}_Z, \mathbf{u})$ as

$$\mathcal{A}(\mathcal{P}_Z, \mathbf{u}) = \bigcap_{i=1}^{m} \mathcal{N}(p^i). \tag{8}$$

Properties 1. to 3. are also verified in this general case.

3. AN ALGORITHM TO REDUCE THE APPROXIMATION SET

In this section we provide an algorithm designed to decision aid in discrete multiattribute decision making problems where a two-dimensional vector utility function has been assessed. The basic idea is as follows: Assume that we have two utility efficient strategies p^i and p^j, with expected utilities vectors $E(\mathbf{u}, p^i) = (u_1^i, u_2^i)$ and $E(\mathbf{u}, p^j) = (u_1^j, u_2^j)$, respectively. Suppose that $u_2^j > u_2^i$ and the DM prefers $E(\mathbf{u}, p^i)$ than $E(\mathbf{u}, p^j)$, then the DM would like to have a better utility for u_1^j. Among the improvement directions we shall consider $\alpha = (1, 0)$ because p^j has its best utility in $E(u_2, \cdot)$ and we are maximizing. In this way, and from the assumption on continuity of the vector utility function, we can improve $E(\mathbf{u}, p^j)$ by taking $(u_1^{j*}, u_2^{j*}) = (u_1^j, u_2^j) + \beta\alpha$, $\beta > 0$, such that $(u_1^{j*}, u_2^{j*}) \sim E(\mathbf{u}, p^i)$ ($\sim$ means indifference). Then, from the monotonicity of the utility, we only have to consider the strategies $p \in \mathcal{A} = \mathcal{A}(\mathcal{P}_Z, \mathbf{u})$ that verifies $E(u_1, p) > u_1^{j*}$, i.e., the set $\mathcal{P}_Z$ is reduced to $\mathcal{P}_{Z_*} = \{p \in \mathcal{A} : E(u_1, p) > u_1^{j*}\}$. Similarly, if $u_1^j > u_1^i$, we take $\alpha = (0,1)$ as the improvement direction, being the most preferred strategy the one that verifies $E(u_2,p) > u_2^{j*}$. In this case, the set $\mathcal{P}_Z$ is reduced to $\mathcal{P}_{Z_*} = \{p \in \mathcal{A} : E(u_2,p) > u_2^{j*}\}$.

Now, we show how to obtain the two utility efficient strategies which are necessary on each run of the procedure. Let $\mathcal{P}_Z^h$ be the set of strategies obtained in iteration h. The process *find* $E(\mathbf{u},p^1)$ and *find* $E(\mathbf{u},p^2)$ is conducted by solving P_{12} and P_{21}. In each iteration h, we compute p^1 and p^2 to obtain the approximation set (7). We compare their expected utilities vectors and, consequently, a new set $\mathcal{P}_{Z_*}^h$ is obtained. In the algorithm, we call $p^j(p^k)$ to the best (worst) current strategy between p^1 and p^2. We consider a set N and we add to this set the strategies indifferent to p^j. On the other hand, to *find* $\left(u_1^{k*},u_2^{k*}\right)$ means to calculate $\left(u_1^{k*},u_2^{k*}\right) = E\left(\mathbf{u},p^k\right) + \beta\alpha$, $\beta \in \mathbb{R}^+$, such that $\left(u_1^{k*},u_2^{k*}\right) \sim (u_1^j,u_2^j)$. We will use the auxiliary variable x to keep the superscript of the best strategy in the previous iteration, to know when there is a movement to a better strategy.

The algorithm is:

Step 0. Let $\mathcal{P}_{Z_*}^0 = \mathcal{P}_Z$, $h=0$, $j=1$ and $N=\emptyset$.
Step 1. *Find* $E(\mathbf{u},p^1)$ and $E(\mathbf{u},p^2)$.
Step 2. Let $\mathcal{P}_Z^h = \mathcal{A}(\mathcal{P}_{Z_*}^h,\mathbf{u})$.
Step 3. If $E(\mathbf{u},p^1) = E(\mathbf{u},p^2)$, the most preferred strategies are

$$N = N \cup \{p \in \mathcal{P}_Z^h : E(\mathbf{u},p) = E\left(\mathbf{u},p^1\right)\}$$

and stop. Otherwise, go to next step.
Step 4. If $E(\mathbf{u},p^1) \sim E(\mathbf{u},p^2)$, let $(u_1^{2*},u_2^{2*}) = E(\mathbf{u},p^2)$ (note that we can also take $(u_1^{1*},u_2^{1*}) = E(\mathbf{u},p^1)$) and go to step 9. Otherwise, go to next step.
Step 5. Let $x=j$.
Step 6. If $E(\mathbf{u},p^1) \succ E(\mathbf{u},p^2)$, then
$j=1$ (the best)
- If $x \neq 1$, then $N=\emptyset$
- *Find* (u_1^{2*},u_2^{2*}) and go to step 9

Otherwise, go to next step.
Step 7. If $E(\mathbf{u},p^2) \succ E(\mathbf{u},p^1)$, then
$j=2$ (the best)
– If $x \neq 2$, then $N=\emptyset$
– *Find* (u_1^{1*},u_2^{1*}) and go to next step.

Step 8. Let $N = N \cup \left\{p \in \mathcal{P}_Z^h : E(\mathbf{u},p) = (u_1^{1*},u_2^{1*})\right\}$, determine

$$\mathcal{P}_{Z_*}^{h+1} = \left\{p \in \mathcal{P}_Z^h : E(u_2,p) > u_2^{1*}\right\}$$

and *find* $E(\mathbf{u},p^1)$ in $\mathcal{P}_{Z_*}^{h+1}$. Let $h=h+1$ and go to step 2.

Step 9. Let $N = N \cup \left\{ p \in \mathcal{P}_Z^h : E(\mathbf{u},p) = \left(u_1^{2*}, u_2^{2*}\right) \right\}$, determine

$$\mathcal{P}_{Z_*}^{h+1} = \left\{ p \in \mathcal{P}_Z^h : E(u_1, p) > u_1^{2*} \right\}$$

and *find* $E(\mathbf{u},p^2)$ in $\mathcal{P}_{Z_*}^{h+1}$. Let $h = h+1$ and go to step 2.

The procedure *find* $\left(u_1^{k*}, u_2^{k*}\right)$ may be conducted by means of questions to the DM about two appropriate strategies to converge to an indifferent point. Graphically, this means that the point $E(\mathbf{u},p^k)$ would move in the improvement direction until the intersection with the isocurve through $E(\mathbf{u},p^j)$ in a point $\left(u_1^{k*}, u_2^{k*}\right)$.

Therefore, we have a search-oriented algorithm that will interactively reduce the set of strategies because of the irrevocable DM's responses. The method is not very demanding because the DM only has to answer to qualitative questions he is faced. On the other hand, the computational effort is small. Observe that if we do not consider the procedures *find* $\left(u_1^{k*}, u_2^{k*}\right)$ and calculate $\mathcal{P}_{Z_*}^{h+1}$, we shall have for each iteration an optimization problem (maximize or minimize), except in the first iteration where we have two, one for each component of the expected utility vector. Note that in the method we take the approximation set to the utility efficient set instead of the whole set $\mathcal{P}_Z$, this may considerably reduce the number of strategies to be investigated. Furthermore, if we assume that the DM fulfils the assumptions that lead to have linear preference structure, French (1986), then the above algorithm may be rewritten in another simple way. Before providing the algorithm, we shall explain the process *determine* the information set $I_{M_{h+1}}$.

Let $I_{M_o} = \mathbb{R}^+$, if the DM reveals that $E\left(\mathbf{u}^{M_h}, p^1\right) \succsim E\left(\mathbf{u}^{M_h}, p^2\right)$, $h \geq 0$ ($\succsim$ means more preferred or indifferent to), the information set will be $I_{M_{h+1}}$ with $M_{h+1} = \left(\mathbf{k}^{1(h+1)}, \mathbf{k}^{2(h+1)}\right)^t$ where $\mathbf{k}^{1(h+1)}, \mathbf{k}^{2(h+1)}$ are the generators of the polyhedral cone $\{\lambda \mathbf{k} : \lambda > 0,\ \mathbf{k} \in K\}$, where K is the set of elements $\mathbf{k}$ that verify

$$\begin{aligned} &\mathbf{k}E\left(\mathbf{u}^{M_h}, p^1\right) \geqq \mathbf{k}E\left(\mathbf{u}^{M_h}, p^2\right) \\ &\mathbf{k} \in I_{M_h}. \end{aligned}$$

In case that $E\left(\mathbf{u}^{M_h}, p^2\right) \succsim E\left(\mathbf{u}^{M_h}, p^1\right)$, then elements $\mathbf{k}$ must fulfil

$$\begin{aligned} &\mathbf{k}E\left(\mathbf{u}^{M_h}, p^2\right) \leqq \mathbf{k}E\left(\mathbf{u}^{M_h}, p^1\right) \\ &\mathbf{k} \in I_{M_h}. \end{aligned}$$

The algorithm is:

Step 0. Let $\mathcal{P}_{Z_*}^0 = \mathcal{P}_Z$, $h = 0$, $j = 1$ and $N = \emptyset$.
Step 1. *Find* $E\left(\mathbf{u}^{M_h}, p^1\right)$ and $E\left(\mathbf{u}^{M_h}, p^2\right)$ on $\mathcal{P}_{Z_*}^h$.
Step 2. Determine $\mathcal{P}_Z^h = \mathcal{A}(\mathcal{P}_{Z_*}^h, \mathbf{u}^{M_h})$.

Step 3. If $E\left(\mathbf{u}^{M_h}, p^1\right) = E\left(\mathbf{u}^{M_h}, p^2\right)$, stop. We have an ideal point and the solutions are $N = N \cup \left\{p \in \mathcal{P}_Z^h : E(\mathbf{u}^{M_h}, p) = E(\mathbf{u}^{M_h}, p_1)\right\}$.
Step 4. Let $x = j$.
Step 5. If $E\left(\mathbf{u}^{M_h}, p^1\right) \sim E\left(\mathbf{u}^{M_h}, p^2\right)$, then $N = \{p^2\} \cup N$, $\left(u_1^{2*}, u_2^{2*}\right) = E\left(\mathbf{u}^{M_h}, p^2\right)$ and go to step 10.
Step 6. If $E\left(\mathbf{u}^{M_h}, p^1\right) \succ E\left(\mathbf{u}^{M_h}, p^2\right)$, then $j = 1$ and $k = 2$. Otherwise, $j = 2$ and $k = 1$.
Step 7. If $j \neq x$, then $N = \emptyset$.
Step 8. Let $\mathcal{P}_{Z*}^{h+1} = \mathcal{P}_Z^h$ and $h = h + 1$.
Step 9. *Determine* I_{M_h} and go to step 1.
Step 10. Calculate

$$\mathcal{P}_{Z_*}^{h+1} = \left\{p \in \mathcal{P}_Z^h : E(u_1^{M_h}, p) > u_1^{2*}\right\}$$

and *find* $E\left(\mathbf{u}^{M_h}, p^2\right)$ on $\mathcal{P}_{Z_*}^{h+1}$. Let $I_{M_{h+1}} = I_{M_h}$, $h = h + 1$ and go to step 2.

Note that this algorithm is used when the true (but unknown) scalar utility function can be written as a linear combination of the components in u, which is a particular case of the former one.

4. THE GENERAL CASE

In the general case of a vector utility function $\mathbf{u}(\cdot)$ with $m \geqq 3$ components, the set of improvement directions will be included in

$$D = \left\{(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m : \alpha_i \geqq 0, \sum_{i=1}^{m} \alpha_i = 1\right\}. \tag{9}$$

If one strategy p^j has their best values in the expected utility vector $E\left(\mathbf{u}, p^j\right)$ for coordinates $i_1, \ldots, i_r$, we take as improvement directions the subset

$$D_j = \{\alpha \in D : \alpha_{i_1} = 0, \ldots, \alpha_{i_r} = 0\}.$$

For example, we can take as improvement direction the vector $\alpha \in D_j$ such that $\alpha_i = 1/(m - r)$, $i \neq i_1, \ldots, i_r$, or simply, ask DM to suggest improvement directions in D_j, to conduct the process *find* $\left(u_1^{k*}, \ldots, u_m^{k*}\right)$ in the general case. The procedure *find* $E\left(\mathbf{u}, p^j\right)$ is led by solving problem $P_{j,j+1,\ldots,m,1,\ldots,j-1}$.

Procedure *modify* $\mathcal{P}_Z^h$ *due to* $\left(u_1^{k*}, \ldots, u_m^{k*}\right)$ is analogous to the case $m = 2$ and now it means that $\mathcal{P}_{Z_*}^h = \mathcal{P}_Z^h - \left\{p \in \mathcal{P}_Z^h : E\left(u_i, p\right) \leqq u_i^{k*},\ i \neq i_1, \ldots, i_r\right\}$. The best solution in the current iteration will be $E\left(\mathbf{u}, p^t\right)$ and Y will remain as the set of indifferent solutions to the current best solution.

Now, let I be the set that contains the superscripts of the strategies that provides the same expected utility than p^t. The algorithm is as follows:

Step 0. Let $\mathcal{P}_{Z_*}^0 = \mathcal{P}_Z$, $h = 0$, $t = 1$, $Y = \emptyset$ and $I = \emptyset$.

Step 1. For all $l \in \{1, \ldots, m\} - I$, *find* $E\left(\mathbf{u}, p^l\right)$ in $\mathcal{P}^h_{Z_*}$.

Step 2. Determine $\mathcal{P}^h_Z = \mathcal{A}\left(\mathcal{P}^h_{Z_*}, \mathbf{u}\right)$.

Step 3. If all $E(\mathbf{u}, p^l)$ were equals, we would have the most preferred strategies in

$$Y \cup \left\{p \in \mathcal{P}^h_Z : E\left(\mathbf{u}, p\right) = E(\mathbf{u}, p^l)\right\}$$

and stop. Otherwise, let $x = t$ and go to next step.

Step 4. Calculate the minimum t that fulfils $E\left(\mathbf{u}, p^t\right) \succ E\left(\mathbf{u}, p^l\right)$, for all l and let $I = \emptyset$.

Step 5. If $E\left(\mathbf{u}, p^t\right) \neq E\left(\mathbf{u}, p^x\right)$, then $Y = \emptyset$. Otherwise, go to next step.

Step 6. From $j = 1$ to m :

- *Find* $E\left(\mathbf{u}, p^j\right)$;
- If $E\left(\mathbf{u}, p^j\right) = E\left(\mathbf{u}, p^t\right)$, then $I = I \cup \{j\}$;
- If $E\left(\mathbf{u}, p^j\right) \sim E\left(\mathbf{u}, p^t\right)$, then

$$Y = Y \cup \left\{p \in \mathcal{P}^h_Z : E\left(\mathbf{u}, p\right) = E\left(\mathbf{u}, p^j\right)\right\}$$

  and *modify* $\mathcal{P}^h_Z$ *due to* $\left(u^{j*}_1, \ldots, u^{j*}_m\right) = E\left(\mathbf{u}, p^j\right)$;
- Otherwise, *find* $\left(u^{k*}_1, \ldots, u^{k*}_m\right)$ let

$$Y = Y \cup \left\{p \in \mathcal{P}^h_Z : E\left(\mathbf{u}, p\right) = \left(u^{k*}_1, \ldots, u^{k*}_m\right)\right\}$$

  and *modify* $\mathcal{P}^h_Z$ *due to* $\left(u^{k*}_1, \ldots, u^{k*}_m\right)$;
- End.

Step 7. Let $\mathcal{P}^{h+1}_{Z_*} = \mathcal{P}^h_{Z_*}$, $h = h + 1$, and go to step 1.

Clearly, this algorithm has same advantages than the one for two components. As above, we can also consider a general algorithm for the case of linear preference structure.

5. AN APPLICATION TO UNIVERSITY SELECTION

In this section we present an example about university selection to illustrate the algorithms in the case of a vector utility function with two components. However, we want to emphasize that the more components in the vector utility function the more useful is the algorithm.

Let us consider a student (DM) selecting a university for the next year. He uses four attributes, which corresponds to the lowest-level objectives of his hierarchy tree. Thus, each university is characterized by a four dimensional attribute vector $\mathbf{z} = (z_1, z_2, z_3, z_4)$, where

z_1 – registration fee ($\times 10^4$ Spanish ptas),
z_2 – distance from university to family residence ($\times 10$ km),
z_3 – distance from university to accommodation (in km),
z_4 – accommodation cost ($\times 5 \cdot 10^4$ ptas).

However, the registration fee as well as the accommodation cost, depend on the Consumer Price Index (CPI) which may increase or remain. The experts assert that the CPI will remain with probability of .7 and will increase with probability .3.

We have grouped z_1 and z_4 (costs) and, on the other hand, z_2 and z_3 (distance). We want to minimize them all. The student reveals that it is more important to save money in accommodation than in registration, because he knows that registration fee is proportional to education quality. He considers six times more important a reduction in accommodation cost than in registration fee. Moreover, he considers eight times more important to reduce the distance from university to accommodation than to family residence. Such information is modelled by means of a vector utility function, where the components are $u'_1(\mathbf{z}) = z_1 + 6z_4$ and $u'_2(\mathbf{z}) = z_2 + 8z_3$. We have to minimize $E\left(\mathbf{u},p^j\right)$ over $\mathcal{P}_Z = \left\{p^1, \ldots, p^{26}\right\}$, where we identify, for instance, strategy p^1 with lottery

$$\begin{pmatrix} .7 & .3 \\ (10,10,1,25) & (11,10,1,27) \end{pmatrix}$$

and so on (see Table 1). We can transform both components of $\mathbf{u}'$ to have $\mathbf{u}\left(\mathbf{z}\right) = -\mathbf{u}'\left(\mathbf{z}\right)$, and thus a maximization problem. Furthermore, if we normalize the weights, the vector utility function will be

$$\mathbf{u}(\mathbf{z}) = \left(-\tfrac{1}{7}z_1 - \tfrac{6}{7}z_4, -\tfrac{1}{9}z_2 - \tfrac{8}{9}z_3\right).$$

Table 1 presents twenty six strategies p^i, one of them must be selected by the DM. Such table also presents the respective consequences $\mathbf{z} = (z_1, z_2, z_3, z_4)$ and utility vectors $\mathbf{u} = (u_1, u_2)$. Table 2 shows the expected utility vectors $E\left(\mathbf{u},p^j\right)$ for each strategy.

In step 1 we obtain by applying the first algorithm to our selection problem that (see Table 2))

$$E\left(\mathbf{u},p^{14}\right) = (-.399, -.320) \quad \text{and} \quad E\left(\mathbf{u},p^1\right) = (-.842, -.052).$$

and in step 2, the approximation set

$$\mathcal{P}_Z^0 = \mathcal{A}(\mathcal{P}_{Z*}^0, \mathbf{u}) = \left\{p^1, p^2, p^5, p^{12}, p^{13}, p^{14}, p^{15}, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}\right\}$$

which contains 13 strategies.

We present $E\left(\mathbf{u},p^{14}\right)$ and $E\left(\mathbf{u},p^1\right)$ to the DM, who has three options:

(1) $E\left(\mathbf{u},p^1\right) \sim E\left(\mathbf{u},p^{14}\right)$ (Step 4). Then, we obtain

$$\mathcal{P}_{Z_*}^1 = \{p^2, p^5, p^{12}, p^{13}, p^{14}, p^{15}, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}\}$$

and (Step 9) $E\left(\mathbf{u},p^{15}\right) = (-.425, -.080)$. We come back to Step 2 and we obtain

$$\mathcal{P}_Z^1 = \mathcal{A}(\mathcal{P}_{Z_*}^1, \mathbf{u}) = \{p^{14}, p^{15}\}.$$

The set of strategies has been reduced to only two, one of them must be chosen by the DM.

Table 1. The set of strategies with their consequences and utility vectors.

p^j	Remains Prob. = .7	Increases Prob. = .3	p^j	Remains Prob. = .7	Increases Prob. = .3
p^1	$(10,10,1,25)$ $(-.830,-.052)$	$(11,10,1,27)$ $(-.868,-.052)$	p^{14}	$(23,64,3,10)$ $(-.353,-.320)$	$(25,64,3,15)$ $(-.502,-.320)$
p^2	$(11,50,5,15)$ $(-.504,-.261)$	$(13,50,5,17)$ $(-.553,-.261)$	p^{15}	$(24,15,2,12)$ $(-.420,-.080)$	$(25,15,2,13)$ $(-.439,-.080)$
p^3	$(12,100,3,18)$ $(-.603,-.494)$	$(17,100,3,21)$ $(-.684,-.494)$	p^{16}	$(25,18,1,16)$ $(-.552,-.090)$	$(27,18,1,16)$ $(-.536,-.090)$
p^4	$(13,120,2,16)$ $(-.539,-.587)$	$(15,120,2,17)$ $(-.535,-.587)$	p^{17}	$(26,24,6,18)$ $(-.618,-.139)$	$(30,24,6,20)$ $(-.666,-.139)$
p^5	$(14,60,4,14)$ $(-.474,-.305)$	$(15,60,4,19)$ $(-.618,-.305)$	p^{18}	$(27,29,4,19)$ $(-.652,-.155)$	$(30,29,4,20)$ $(-.666,-.155)$
p^6	$(15,40,6,30)$ $(-1.0,-.216)$	$(16,40,6,31)$ $(-1.0,-.216)$	p^{19}	$(28,33,5,17)$ $(-.588,-.178)$	$(28,33,5,17)$ $(-.569,-.178)$
p^7	$(16,200,9,25)$ $(-.837,-1.0)$	$(18,200,9,27)$ $(-.875,-1.0)$	p^{20}	$(29,34,3,21)$ $(-.720,-.175)$	$(30,34,3,22)$ $(-.729,-.175)$
p^8	$(17,154,1,28)$ $(-.937,-.747)$	$(20,154,1,30)$ $(-.972,-.747)$	p^{21}	$(30,42,2,23)$ $(-.787,-.210)$	$(31,42,2,24)$ $(-.794,-.210)$
p^9	$(18,160,5,26)$ $(-.872,-.791)$	$(21,160,5,27)$ $(-.878,-.791)$	p^{22}	$(31,157,1,25)$ $(-.854,-.762)$	$(31,157,1,25)$ $(-.825,-.762)$
p^{10}	$(19,200,8,24)$ $(-.808,-.996)$	$(25,200,8,25)$ $(-.819,-.996)$	p^{23}	$(32,199,5,28)$ $(-.953,-.980)$	$(35,199,5,30)$ $(-.988,-.980)$
p^{11}	$(20,145,7,17)$ $(-.579,-.727)$	$(21,145,7,25)$ $(-.815-.727)$	p^{24}	$(33,165,7,26)$ $(-.888,-.823)$	$(35,165,7,30)$ $(-.988,-.823)$
p^{12}	$(21,45,5,11)$ $(-.384,-.236)$	$(22,45,5,16)$ $(-.531,-.236)$	p^{25}	$(34,187,9,23)$ $(-.791,-.937)$	$(35,187,9,25)$ $(-.830,-.937)$
p^{13}	$(22,39,4,15)$ $(-.516,-.203)$	$(25,39,4,20)$ $(-.661,-.203)$	p^{26}	$(35,145,8,11)$ $(-.399,-.731)$	$(37,145,8,12)$ $(-.420,-.731)$

Table 2. Expected utility vectors.

p^j	$E\left(\mathbf{u},p^j\right)$	p^j	$E\left(\mathbf{u},p^j\right)$
p^1	$(-.842,\ -.052)$	p^{14}	$(-.399,\ -.320)$
p^2	$(-.519,\ -.261)$	p^{15}	$(-.425,\ -.080)$
p^3	$(-.628,\ -.494)$	p^{16}	$(-.547,\ -.090)$
p^4	$(-.544,\ -.587)$	p^{17}	$(-.633,\ -.139)$
p^5	$(-.518,\ -.305)$	p^{18}	$(-.657,\ -.155)$
p^6	$(-1.0,\ -.216)$	p^{19}	$(-.582,\ -.178)$
p^7	$(-.849,\ -1.0)$	p^{20}	$(-.723,\ -.175)$
p^8	$(-.948,\ -.747)$	p^{21}	$(-.789,\ -.210)$
p^9	$(-.874,\ -.791)$	p^{22}	$(-.845,\ -.762)$
p^{10}	$(-.811,\ -.996)$	p^{23}	$(-.964,\ -.926)$
p^{11}	$(-.652,\ -.727)$	p^{24}	$(-.919,\ -.823)$
p^{12}	$(-.429,\ -.236)$	p^{25}	$(-.803,\ -.937)$
p^{13}	$(-.560,\ -.203)$	p^{26}	$(-.405,\ -.731)$

(2) $E\left(\mathbf{u},p^{14}\right) \succ E\left(\mathbf{u},p^{1}\right)$ (Step 6). Then, we have to determine $\left(u_1^{2*},u_2^{2*}\right) \sim E(\mathbf{u},p^{14})$. Suppose the DM considers $\left(u_1^{2*},u_2^{2*}\right) = (-.690,-.052)$.

Therefore,

$$\mathcal{P}^1_{Z_*} = \{p^2, p^5, p^{12}, p^{13}p^{14}, p^{15}, p^{16}, p^{17}, p^{18}, p^{19}, p^{21}\}$$

(Step 9). We go back to Step 2 and obtain

$$\mathcal{P}^1_Z = \mathcal{A}(\mathcal{P}^1_{Z_*}, \mathbf{u}) = \{p^{14}, p^{15}\}.$$

Observe that we have same solution as above. In case that u_1^{2*} were greater than $-.425$, then the most preferred solution would have been p^{14}.

(3) $E\left(\mathbf{u},p^1\right) \succ E\left(\mathbf{u},p^{14}\right)$ (Step 7). Then, we have to determine $\left(u_1^{1*}, u_2^{1*}\right) \sim E\left(\mathbf{u},p^1\right)$. Assume that DM considers $\left(u_1^{1*}, u_2^{1*}\right) = (-.399, -.1)$. Hence, $\mathcal{P}^1_{Z_*} = \{p^1, p^{15}, p^{16}\}$ (Step 8). Coming back to step 2, we obtain

$$\mathcal{P}^1_Z = \mathcal{A}(\mathcal{P}^1_{Z_*}, \mathbf{u}) = \{p^1, p^{15}\} \ .$$

We have arrived to situation analogous to the above two cases. In the event that u_2^{1*} were greater than $-.080$, then the most preferred solution would be p^1.

If the DM assumes the axioms to have a scalar utility function as a linear combination of the components of the vector utility function, we can use the information provided by the DM in steps 6 (or 7), where he revealed that $E\left(\mathbf{u},p^{14}\right)$ is preferred to $E\left(\mathbf{u},p^1\right)$ (or $E\left(\mathbf{u},p^1\right)$ is preferred to $E\left(\mathbf{u},p^{14}\right)$), to construct the generators of the information set. In fact, if we have the first assertion, the weights $\mathbf{k} = (k_1, k_2)$ of the true DM's scalar utility function, have to verify

$$\begin{gathered}\mathbf{k}E\left(\mathbf{u},p^{14}\right) \geqq \mathbf{k}E\left(\mathbf{u},p^1\right)\\ \mathbf{k} \in I_{M_0}\end{gathered}$$

and define a new information set I_{M_1}, where

$$M_1 = \begin{pmatrix} 1 & 0 \\ .377 & .623 \end{pmatrix}.$$

Hence, the associated vector utility function is

$$\mathbf{u}^{M_1}(\mathbf{z}) = (-.143z_1 - .857z_4, -.054z_1 - .069z_2 - .554z_3 - .323z_4).$$

Thus, the approximation set for $\mathbf{u}^{M_1}(\cdot)$ (step 2) is

$$\mathcal{P}^1_Z = \mathcal{A}(\mathcal{P}^1_{Z_*}, \mathbf{u}^{M_1}) = \{p^{14}, p^{15}\}.$$

On the other hand, if the DM reveals that $E\left(\mathbf{u},p^1\right) \succ E\left(\mathbf{u},p^{14}\right)$, the weights $\mathbf{k}$ have to verify

$$\begin{gathered}\mathbf{k}E\left(\mathbf{u},p^1\right) \geqq \mathbf{k}E\left(\mathbf{u},p^{14}\right)\\ \mathbf{k} \in I_{M_0}\end{gathered}$$

i.e., the information set is I_{M_1}, where

$$M_1 = \begin{pmatrix} .377 & .623 \\ 0 & 1 \end{pmatrix}.$$

Now, the associated vector utility function is

$$\mathbf{u}^{M_1}(\mathbf{z}) = (-.054z_1 - .069z_2 - .554z_3 - .323z_4, -.125z_2 - .878z_3).$$

The approximation set for $\mathbf{u}^{M_1}(\cdot)$ (Step 2) is

$$\mathcal{P}_Z^1 = \mathcal{A}(\mathcal{P}_{Z_*}^1, \mathbf{u}^{M_1}) = \{p^1, p^{15}\}.$$

We note that the solutions obtained applying this algorithm are, in this case, the same than those obtained with the first one. However, the number of iterations usually will be smaller, although in this last method it is necessary for the DM to check the respective assumptions, which might be difficult to verify.

6. CONCLUSIONS

In multiattribute decision making under risk, the utility efficient set plays an important role in the solution process. Its generation may be very difficult and, moreover, it may be too extensive for the DM to make an easy choice. We propose an interactive approach to overcome such drawbacks, which uses an idea of approximation to the utility efficient set. Then, we reduce the set of strategies of interest based on information revealed by the DM from comparison of pairs of strategies. The procedure stops when a single strategy is achieved or a reduced enough set of strategies for the DM is obtained. We also consider a second procedure that is a variant of first one, valid for the case in which the DM has a linear utility function obtained as a combination of the components of the vector utility function.

It is a comfortable procedure for the DM, because he only has to answer qualitative questions for comparison of each pair of strategies. Furthermore, from a computational point of view, it is only necessary to solve as much optimization problems as the number of components of the DM's vector utility function. Finally, an application to university selection shows the power of the procedure.

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