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WEIGHTED QUADRATURE FORMULAE OF GAUSS–CHRISTOFFEL–STANCU TYPE

DIMITRIE D. STANCU,* IOANA TAŞCU† and ALINA BEIAN-PUTURA‡

Abstract. In the present paper we consider weighted integrals and develop explicit quadrature formulae of Gauss–Christoffel–Stancu type using simple Gaussian nodes and multiple fixed nodes. Given the multiple fixed nodes and their multiplicities, we present some algorithms for finding the Gaussian nodes, the coefficients and the remainders of the corresponding quadrature formulae. Several illustrative examples are presented in the case of some classical weight functions.

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1. INTRODUCTION

Let w be a nonnegative weight function assumed integrable over a bounded or unbounded interval (a, b) of the real axis. We require that all the moments of this weight function

$$c_k = \int_a^b x^k w(x)\mathrm{d}x, \quad k = 0, 1, 2, \dots$$

exist and $c_0 > 0$.

Integrands in which such a weight function is present, as a multiplicative factor, appear frequently in the theory of orthogonal families of functions.

Let f be a real-valued function having continuous derivatives of whatever orders will be needed.

We suppose that we want to construct quadrature formulae, for weighted integrals, of the following form

$$U(f) = F(f) + R(f), \tag{1}$$

where

$$U(f) = U(w; f) = \int_a^b w(x)f(x)\mathrm{d}x,$$

$$F(f) = \sum_{k=1}^{m} A_k f(x_k) + \sum_{i=1}^{s} \sum_{j=0}^{r_i - 1} C_{i,j} f^{(j)}(a_i) \tag{2}$$

*"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, str. M. Kogălniceanu 1, 3400 Cluj-Napoca, Romania, e-mail: ddstancu@math.ubbcluj.ro.
†North University, Baia Mare, Romania.
‡Computer Science Highschool, Bistriţa, Romania.

and $R(f) = R(w; f)$, the remainder or the complimentary term, is, by definition, the difference $U(f) - F(f)$.

We denote by u the polynomial of the distinct nodes x_k and by ω the polynomial of the multiple fixed nodes a_i, namely

(3) $$u(x) = (x - x_1)(x - x_2) \dots (x - x_m),$$

(4) $$\omega(x) = (x - a_1)^{r_1} (x - a_2)^{r_2} \dots (x - a_s)^{r_s}.$$

Here $r_1, r_2, \dots, r_s$ are nonnegative integers and a_i are preassigned nodes, such that $\omega(x) \geq 0$ on the interval (a, b).

2. USE OF THE LAGRANGE–HERMITE INTERPOLATION

We will use a method of parameters (see D. D. Stancu [14], [15]) for constructing a general Gauss–Christoffel type quadrature rule by using simple nodes x_k and preassigned multiple nodes a_i.

We shall start from the Lagrange–Hermite interpolation formula corresponding to the function f, to the simple nodes x_k, to the multiple nodes a_i and to other nondetermined simple nodes $t_1, t_2, \dots, t_m$, distinct from the other nodes. This formula has the form

(5) $$f(x) = (L_{2m+p-1} f)(x) + (Q_{2m+p-1} f)(x).$$

The interpolating polynomials

$$(L_{2m+p-1} f)(x) = v(x)(L_H f_1)\Big(x;\ x_k,\ \begin{smallmatrix} a_i \\ r_i \end{smallmatrix}\Big) + \omega(x) u(x) (L_H f_2)(x; t_h),$$

where

$$v(x) = (x - t_1) \dots (x - t_m), \quad f_1 = \tfrac{f}{v}, \quad f_2 = \tfrac{f}{\omega u}, \quad p = r_1 + \dots + r_m$$

and the remainder is expressed by

$$(Q_{2m+p-1} f)(x) = \omega(x) u(x) v(x) \left[\begin{smallmatrix} x & x_k & a_i \\ 1 & 1 & r_i \end{smallmatrix} ; f \right],$$

the square brackets indicating the divided difference of f on the indicating nodes; the numbers beneath the nodes designate their multiplicities.

More explicitly the interpolating polynomial can be written as follows

(6) $$\begin{aligned}(L_{2m+p-1} f)(x) = &\sum_{k=1}^{m} \tfrac{u_k(x)}{u_k(x_k)} \cdot \tfrac{v(x)}{v(x_k)} \cdot \tfrac{\omega(x)}{\omega(x_k)} f(x_k) \\ &+ \sum_{h=1}^{m} \tfrac{u(x)}{u(t_h)} \cdot \tfrac{v_h(x)}{v_h(t_h)} \cdot \tfrac{\omega(x)}{\omega(t_h)} f(t_h) + \\ &+ \sum_{i=1}^{s} \sum_{j=0}^{r_i-1} \sum_{\nu=0}^{r_i-j-1} \tfrac{(x-a_i)^j}{j!} \left[\tfrac{(x-a_i)^\nu}{\nu!} \left(\tfrac{1}{\omega_i(x)} \right)^{(\nu)}_{a_i} \right] \omega_i(x) f^{(j)}(a_i),\end{aligned}$$

where

$$u_k(x) = \frac{u(x)}{(x - a_k)}, \quad v_h(x) = \frac{v(x)}{(x - t_h)}, \quad \omega_i(x) = \frac{\omega(x)}{(x - a_i)^{r_i}}.$$

3. CONSTRUCTION OF THE QUADRATURES BY THE METHOD OF PARAMETERS

If we multiply formula (5) by $w(x)$ and integrate we obtain a quadrature formula of the form

$$\begin{aligned}\int_a^b w(x)f(x)\mathrm{d}x &= \sum_{k=1}^{m} A_k f(x_k) + \sum_{h=1}^{m} B_h f(t_h)\\ &\quad + \sum_{i=1}^{s}\sum_{j=0}^{r_i-1} C_{i,j} f^{(j)}(a_i) + R(f),\end{aligned} \tag{7}$$

where

$$R(f) = R(w;f) = \int_a^b w(x)u(x)v(x)\omega(x)\left[\begin{smallmatrix} x & x_k & t_h & a_i \\ 1 & 1 & 1 & r_i \end{smallmatrix}; f\right]. \tag{8}$$

Because the divided difference which occurs in (8) is of order $2m+p$, it follows that the quadrature formula (7) has the degree of exactness $N = 2m+p-1$.

Now we want to determine the nodes x_k so that we have $B_1 = B_2 = \cdots = B_m = 0$, for any values of the parameters $t_1, t_2, \ldots, t_m$.

Since the coefficients B_h are given by the formula

$$B_h = \int_a^b w(x)\tfrac{u(x)}{u(t_h)} \cdot \tfrac{v_h(x)}{v_h(t_h)} \cdot \tfrac{\omega(x)}{\omega(t_h)}\mathrm{d}x$$

and t_h are arbitrary, it follows that $B_h = 0$, $h = 1, 2, \ldots, m$, if and only if the polynomial $u(x)$ is orthogonal on (a,b), with respect to the weight function $w\omega$, to any polynomial of degree $m-1$. Consequently the nodes $x_1, x_2, \ldots, x_m$ should be the m real and distinct roots of the polynomial of Christoffel–Szegö, defined by the formula

$$(9)\qquad U_m(x) = \frac{1}{\omega(x)} \begin{vmatrix} P_m(x) & P_{m+1}(x) & \dots & P_{m+p}(x) \\ P_m(a_1) & P_{m+1}(a_1) & \dots & P_{m+p}(a_1) \\ P'_m(a_1) & P'_{m+1}(a_1) & \dots & P'_{m+p}(a_1) \\ \vdots & \vdots & & \vdots \\ P_m^{(r_1-1)}(a_1) & P_{m+1}^{(r_1-1)}(a_1) & \dots & P_{m+p}^{(r_1-1)}(a_1) \\ P_m(a_2) & P_{m+1}(a_2) & \dots & P_{m+p}(a_2) \\ \vdots & \vdots & & \vdots \\ P_m(a_s) & P_{m+1}(a_s) & \dots & P_{m+p}(a_s) \\ P'_m(a_s) & P'_{m+1}(a_s) & \dots & P'_{m+p}(a_s) \\ \vdots & \vdots & & \vdots \\ P_m^{(r_s-1)}(a_s) & P_{m+1}^{(r_s-1)}(a_s) & \dots & P_{m+p}^{(r_s-1)}(a_s) \end{vmatrix},$$

where $\{P_n\}$ is the orthogonal family of polynomials on (a,b), with respect to the weight function w.

We mention that formula (9) was given by E. B. Christoffel [1] in the case $w(x)=1$, $r_1=r_2=\cdots=r_s=1$ and by G. Szegö [18] in the case $w(x)=1$ and arbitrary $r_1,r_2,\dots,r_s$.

As a consequence, if x_k are the roots of the polynomial (9) then we get the following quadrature formula

$$(10)\qquad \int_a^b w(x)f(x)\mathrm{d}x = \sum_{k=1}^m A_k f(x_k) + \sum_{i=1}^s \sum_{j=0}^{r_i-1} C_{i,j} f^{(j)}(a_i) + R(f).$$

It should be remarked that x_k (the **Gaussian nodes**) can be found also by determining the relative minimum of the following function of m variables

$$G(u_1,\dots,u_m) = \int_a^b w(x)(x-u_1)^2\dots(x-u_m)^2\mathrm{d}x.$$

4. DETERMINATION OF THE COEFFICIENTS AND THE REMAINDER

Because $t_1,t_2,\dots,t_m$ are arbitrary numbers, we can make $t_k\to x_k$, $k=1,2,\dots,m$, and we are able to see that we have

$$(11)\qquad A_k = \int_a^b w(x)\left(\tfrac{u_k(x)}{u_k(x_k)}\right)^2 \tfrac{\omega(x)}{\omega(x_k)}\mathrm{d}x$$

and

$$(12)\qquad R(f) = \int_a^b w(x)u^2(x)\omega(x)\left[x \;\begin{smallmatrix} x_k \\ 2\end{smallmatrix}\; \begin{smallmatrix} a_i \\ r_i\end{smallmatrix}; f\right]\mathrm{d}x,$$

where $k=1,2,\dots,m$ and $i=1,2,\dots,s$.

We can see from (11) that the coefficients A_k are all positive, but the coefficients $C_{i,j}$ are not necessarily positive.

According to the interpolation formula (6) these last coefficients can be expressed by the formula

$$C_{i,j} = \int_a^b w(x)\tfrac{(x-a_i)^j}{j!}\left[\tfrac{(x-a_i)^\nu}{\nu!}\left(\tfrac{1}{\omega_i(x)}\right)^{(\nu)}_{a_i}\right]\omega_i(x)\mathrm{d}x.$$

If we assume that $f \in C^{2m+p}(a,b)$, by using the mean-value theorem of divided differences we can write the following representation of the remainder

$$R(f) = \tfrac{f^{(2m+p)}(\xi)}{(2m+p)!}\int_a^b w(x)u^2(x)\omega(x)\mathrm{d}x, \tag{13}$$

for a certain point $\xi \in (a,b)$.

Remark. In some special cases formula (9) can be simplified. For instance if $a=-b$, $w(x)$ and $\omega(x)$ are even in $(-b,b)$ and the fixed nodes are $\pm a_1, \pm a_2, \dots, \pm a_q$, $2q=r$, having the orders of multiplicity the even numbers $r_1, r_2, \dots, r_q$, then the determinant from (9) reduces to a determinant having the first row formed by the following elements $P_m(x), P_{m+2}(x), \dots, P_{m+r}(x)$ and the next rows are $P_m^{(j)}(a_i), P_{m+2}^{(j)}(a_i), \dots, P_{m+r}^{(j)}(a_i)$, where $i=1,2,\dots,q$; $j=0,1,\dots,r_i-1$. $\square$

5. SPECIAL CASES OF FORMULA (7)

1) If $-a_1 = a_2 = b$ $(p=2)$, $(a,b)=(-b,b)$ and $w(x)$ is an even function then we obtain the quadrature formula

$$\begin{aligned}\int_{-b}^b w(x)f(x)\mathrm{d}x = &\sum_{k=1}^m A_k f(x_k) + \sum_{j=0}^{r_1-1} B_j f^{(j)}(-b) + \sum_{h=0}^{r_2-1} C_h f^{(h)}(b)\\ &+ \tfrac{f^{(2m+r_1+r_2)}(\xi)}{(2m+r_1+r_2)!}\int_{-b}^b w(x)(x+b)^{r_1}(x-b)^{r_2}u^2(x)\mathrm{d}x.\end{aligned} \tag{14}$$

We can see that when $r_1 = r_2 = r$ we have $B_j = C_j$ if j is even and $B_j = -C_j$ if j is odd. In this case the preceding formula becomes

$$\begin{aligned}\int_{-b}^b w(x)f(x)\mathrm{d}x = &\sum_{k=1}^m A_k f(x_k) + \sum_{j=0}^{r-1} B_j[f^{(j)}(-b) + (-1)^j f^{(j)}(b)]\\ &+ \tfrac{f^{(2m+2r)}(\xi)}{(2m+2r)!}\int_{-b}^b w(x)(x^2-b^2)^r u^2(x)\mathrm{d}x,\end{aligned}$$

where $B_j > 0$, $j=0,1,\dots,r-1$.

2) When $(a,b)=(-b,b)$, $-a_1=a_3=b$, $a_2=0$, $r_1=r_3=r$, $r_2=2s$, $m=2k$ and $w(x)$ is an even function we obtain the quadrature formula

$$\begin{aligned}\int_{-b}^{b} w(x)f(x)\mathrm{d}x = & \sum_{i=1}^{2s} A_i f(x_i) + \sum_{j=0}^{r-1} B_j \left[f^{(j)}(-b)+(-1)^j f^{(j)}(b)\right] \\ & + \sum_{j=0}^{s-1} C_{2j} f^{(2j)}(0) \\ & + \tfrac{f^{(4k+2r+2s)}(\xi)}{(4k+2r+2s)!} \int_{-b}^{b} w(x)x^{2s}(x^2-b^2)^{2r}u^2(x)\mathrm{d}x.\end{aligned} \tag{15}$$

Here the coefficients of $f^{(i)}(0)$ are zero if i is odd.

When the number of Gaussian nodes m is odd: $2k+1$, then one of these will coincide with zero and we get a quadrature formula similar with the preceding one with the multiplicity of the fixed node a_2 increased by two.

3) When the polynomial of the fixed nodes is $\omega(x)=x^{2s}$, $a=-b$, $m=2n$ and the weight function is even, then we can obtain a quadrature formula of the following form

$$\int_{-a}^{a} w(x)f(x)\mathrm{d}x = \sum_{k=1}^{2n} A_k f(x_k) + \sum_{i=0}^{s-1} B_{2i} f^{(2i)}(0) + R(f), \tag{16}$$

because among the fixed nodes occurs also the point $x=0$.

The remainder has the expression

$$R(f)=\tfrac{f^{(4n+2s)}(\xi)}{(4n+2s)!}\int_{-a}^{a} w(x)x^{2s}u^2(x)\mathrm{d}x.$$

The nodes x_k are the roots of the orthogonal polynomial $D_{2n,2s}(x)$ corresponding to the weight function $w(x)x^{2s}$ and to the bounded or unbounded interval $(-a,a)$ (see D. D. Stancu [13]); $u(x)=\widetilde{D}_{2n,2s}(x)$ is with leading coefficient 1.

For the coefficients A_k one finds the following expressions

$$A_k=\int_{-a}^{a} w(x)\Big(\tfrac{D_{2n,2s}(x)}{(x-x_k)D'_{2m,2s}}\Big)^2 \tfrac{x^{2s}}{x_k^{2s}}\mathrm{d}x.$$

Consequently all the coefficients A_k of the quadrature formula (16) are positive.

If we take into account that $D_{2n,2s}(x)$ is a symmetrical polynomial with respect to $w(x)$ and the interval $(-a,a)$ and we assume that we have $x_1<x_2<\cdots<x_{2n}$, we can see that we have

$$A_k=A_{2n-k+1},\quad k=1,2,\ldots,2n.$$

By applying the Christoffel–Darboux formula from the theory of orthogonal polynomials, we can deduce the following explicit expressions for the coefficients A_k (see D. D. Stancu [13]):

$$A_k = \frac{\Delta_{2n-2,2s+2}}{\Delta_{2n-4,2s+2}} \cdot \frac{1}{x_i^{2s}\widetilde{D}'_{2n,2s}(x_i)\widetilde{D}_{2n-1,2s}(x_i)},$$

where $\widetilde{D}_{2n,2s}(x) = x^{2n} + \dots$ and by using the moments of $w(x)$ we have

$$\Delta_{2n,2s} = \begin{vmatrix} c_{2s} & c_{2s+2} & \cdots & c_{2s+2n} \\ c_{2s+2} & c_{2s+4} & \cdots & c_{2s+2n+2} \\ \vdots & \vdots & & \vdots \\ c_{2s+2n} & c_{2s+2n+2} & \cdots & c_{2s+4n} \end{vmatrix}.$$

6. ILLUSTRATIVE EXAMPLES IN THE CASE OF SOME CLASSICAL WEIGHT FUNCTIONS

A) In the case of the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$, where $\alpha, \beta > -1$, we have the orthogonal polynomial of Jacobi

$$J_m^{(\alpha,\beta)}(x) = \tfrac{(-1)^m}{2^m \cdot m!} \cdot \tfrac{1}{\omega(x)}\left[(1-x)^{m+\alpha}(1+x)^{m+\beta}\right]^{(m)}.$$

For the calculation of the integral

$$U(f) = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)\mathrm{d}x$$

we construct some Gauss–Christoffel quadrature formulae.

If the polynomial of the fixed nodes is $\omega(x) = 1 - x^2$, because $r_1 = r_2 = 1$, formula (9) leads us to the solution of the equation

$$\begin{aligned}(\alpha+m+1)(\beta+m+1)(\alpha+\beta+2m+4)J_m^{(\alpha,\beta)}(x)+ \\ +(\alpha-\beta)(\alpha+\beta+2m+3)(m+1)J_{m+1}^{(\alpha,\beta)}(x)- \\ -(\alpha+\beta+2m+2)(m+1)(m+2)J_{m+2}^{\alpha,\beta}(x) = 0.\end{aligned}$$

In the case $m = 1$ we find the Gaussian node $x_1 = \frac{\beta-\alpha}{\alpha+\beta+4}$ and the following Gauss–Christoffel quadrature formula

$$\begin{aligned}&\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta f(x)\mathrm{d}x = \\ &= 2^{\alpha+\beta+1}\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\alpha+2)(\beta+2)\Gamma(\alpha+\beta+4)}\Big[(\alpha+1)(\alpha+2)^2 f(-1)+ \\ &\quad + (\alpha+1)(\beta+1)(\alpha+\beta+4)^2 f\left(\tfrac{\beta-\alpha}{\alpha+\beta+4}\right) + (\beta+1)(\beta+2)^2 f(1)\Big]- \\ &\quad - 2^{\alpha+\beta+2}\tfrac{\Gamma(\alpha+3)\Gamma(\beta+3)}{3(\alpha+\beta+4)\Gamma(\alpha+\beta+6)} f^{(4)}(\xi),\end{aligned}$$

which was first discovered in 1958 by D. D. Stancu [14] (see also [15]).

In the particular case $\alpha = \beta = 0$, it reduces to the known Cavalieri–Simpson formula.

If $\alpha=\frac{1}{2}$, $\beta=-\frac{1}{2}$, we obtain the following quadrature formula

$$\int_{-1}^{1}\sqrt{\tfrac{1-x}{1+x}}f(x)\mathrm{d}x=\tfrac{\pi}{60}\Big[25f(-1)+32f\left(-\tfrac{1}{4}\right)+3f(1)\Big]-\tfrac{\pi}{256}f^{(4)}(\xi).$$

If we consider that the polynomial of the fixed nodes is $\omega(x)=(1-x^2)^2$, then the Gaussian nodes can be found by solving the equation

$$\omega(x)U_m(x)=\begin{vmatrix} J_m(x) & J_{m+2}(x) & J_{m+4}(x)\\ J_m(1) & J_{m+2}(1) & J_{m+4}(1)\\ J_m'(1) & J_{m+2}'(1) & J_{m+4}'(1)\end{vmatrix}=0,$$

where we have denoted by $J_m(x)$ the Jacobi polynomial $J_m^{(\alpha,\alpha)}(x)$.

For $m=3$ we have

$$\big[J_5(1)J_7'(1)-J_5'(1)J_7(1)\big]J_3(x)-\big[J_3(1)J_7'(1)-J_3'(1)J_7(1)\big]J_5(x)+$$
$$+\big[J_3(1)J_5'(1)-J_3'(1)J_5(1)\big]J_7(x)=0.$$

Because

$$\begin{aligned}
J_3(1)&=\tfrac{(\alpha+1)(\alpha+2)(\alpha+3)}{6},\\
J_3'(1)&=\tfrac{(\alpha+2)^2(\alpha+3)}{2},\\
J_5(1)&=\tfrac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+5)}{120},\\
J_5'(1)&=\tfrac{(\alpha+2)(\alpha+3)^2(\alpha+4)(\alpha+5)}{24},\\
J_7(1)&=\tfrac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)(\alpha+6)(\alpha+7)}{5040},\\
J_7'(1)&=\tfrac{(\alpha+2)(\alpha+3)(\alpha+4)^2(\alpha+5)(\alpha+6)(\alpha+7)}{720},
\end{aligned}$$

we find that the corresponding Gaussian nodes are

$$x_1=-\sqrt{\tfrac{3}{2\alpha+9}},\quad x_2=0,\quad x_3=\sqrt{\tfrac{3}{2\alpha+9}}.$$

In this case we obtain the following Gauss–Christoffel–Stancu quadrature formula

$$\begin{aligned}
&\int_{-1}^{1}(1-x^2)^{\alpha}f(x)\mathrm{d}x=\\
&=2^{2\alpha+3}\tfrac{\Gamma(\alpha+1)\Gamma(\alpha+3)}{3(\alpha+3)\Gamma(2\alpha+8)}\Bigg\{32(\alpha+1)(\alpha+2)(\alpha+3)^3f(0)\\
&\quad+(\alpha+1)(\alpha+2)(2\alpha+9)^3\Big[f\Big(-\sqrt{\tfrac{3}{2\alpha+9}}\Big)+f\Big(\sqrt{\tfrac{3}{2\alpha+9}}\Big)\Big]\\
&\qquad+9(8\alpha^2+45\alpha+57)\big(f(-1)+f(1)\big)\\
&\qquad+9(\alpha+1)(\alpha+3)\big(f'(-1)-f'(1)\big)\Bigg\}+R(f),
\end{aligned}$$

where

$$R(f)=\tfrac{2^{2\alpha+2}\Gamma(\alpha+4)\Gamma(\alpha+6)}{4725(2\alpha+9)\Gamma(2\alpha+12)}f^{(10)}(\xi).$$

In the case $\alpha = -\frac{1}{2}$ we find for the Chebyshev first kind weight function the quadrature formula

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{2400}\Big[438f(-1)+15f'(-1)+512f\Big(-\frac{\sqrt{6}}{4}\Big)+500f(0)$$
$$+512f\Big(\frac{\sqrt{6}}{4}\Big)-15f'(1)+438f(1)\Big]+\frac{1}{1238630400}f^{(10)}(\xi).$$

If $\alpha = \frac{1}{2}$ we get for the Chebyshev second kind weight function the quadrature formula

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-x^2}f(x)\mathrm{d}x =$$
$$= \frac{1}{37632}\Big\{6860f(0)+5000\Big[f\Big(-\sqrt{\tfrac{3}{10}}\Big)+f\Big(\sqrt{\tfrac{3}{10}}\Big)\Big]+$$
$$+978[f(-1)+f(1)]+63[f'(-1)-f'(1)]\Big\}+\frac{1}{2654208000}f^{(10)}(\xi).$$

When the polynomial of the fixed nodes is $\omega(x) = x^2$, $m = 3$, and $w(x) = (1-x^2)^\alpha$ then we find a quadrature formula of degree of exactness eleven, namely

$$\int_{-1}^{1}(1-x^2)^\alpha f(x)\mathrm{d}x =$$
$$= \frac{4^{\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+4)}{1225(\alpha+2)\Gamma(\alpha+2)}\Big\{896(\alpha+1)(\alpha+2)^2(34\alpha+123)f(0)$$
$$+2240(\alpha+1)(\alpha+2)^2f''(0)+3(2\alpha+9)\Big[7(\alpha+2)(52\alpha^2+316\alpha+389)$$
$$-(92\alpha^2+396\alpha+179)\Big]\sqrt{7(\alpha+2)(2\alpha+9)}\,[f(x_1)+f(x_5)]$$
$$+3(2\alpha+9)\Big[7(\alpha+2)(52\alpha^2+316\alpha+389)$$
$$+(92\alpha^2+396\alpha+179)\Big]\sqrt{7(\alpha+2)(2\alpha+9)}\,[f(x_2)+f(x_4)]\Big\}$$
$$+\frac{4^{\alpha+1}}{4455(2\alpha+9)(2\alpha+11)}\cdot\frac{\Gamma(\alpha+3)\Gamma(\alpha+7)}{\Gamma(2\alpha+14)}f^{(12)}(\xi),$$

where the Gaussian nodes are

$$-x_1 = x_5 = \sqrt{\frac{7(2\alpha+9)+2\sqrt{7(\alpha+2)(2\alpha+9)}}{(2\alpha+9)(2\alpha+11)}},\quad x_3 = 0,$$
$$-x_2 = x_4 = \sqrt{\frac{7(2\alpha+9)-2\sqrt{7(\alpha+2)(2\alpha+9)}}{(2\alpha+9)(2\alpha+11)}}.$$

In the particular case $\alpha = \frac{1}{2}$ it becomes

$$\int_{-1}^{1} \sqrt{1-x^2} f(x)\mathrm{d}x = \tfrac{\pi}{1568}\Big[392 f(0) + 7f''(0) + 2(49-10\sqrt{7})\big(f(x_1') + f(x_5')\big) +$$

$$+ 2(49+10\sqrt{7})\big(f(x_2') + f(x_4')\big)\Big] + \tfrac{\pi}{2942985830400} f^{(12)}(\xi),$$

where

$$-x_1' = x_5' = \sqrt{\tfrac{7+\sqrt{7}}{12}}, \quad -x_2' = x_4' = \sqrt{\tfrac{7-\sqrt{7}}{12}}.$$

B) In the case of $w(x) = e^{-x^2}$, the interval $(-\infty, \infty)$ and $\omega(x) = x^2$, the Gaussian nodes are given by the equation

$$H_{2m-1}(x) + 2(2m+1)H_{2m-1}(x) = 0,$$

where by $H_n(x)$ we denote the Hermite orthogonal polynomial:

$$H_n(x) = (-1)^n e^{x^2}[e^{-x^2}]^{(n)}.$$

For $m = 3$ we find the Gaussian nodes

$$-x_1 = x_5 = \sqrt{\tfrac{7+\sqrt{14}}{2}}, \quad x_3 = 0, \quad -x_2 = x_4 = \sqrt{\tfrac{7-\sqrt{14}}{2}}$$

and the Gauss–Christoffel–Stancu quadrature formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\mathrm{d}x = \tfrac{\sqrt{\pi}}{4900}\Big[3808 f(0) + 280 f''(0) + 3(91+23\sqrt{14})\big(f(x_2) + f(x_4)\big)$$

$$+ 3(91-23\sqrt{14})\big(f(x_1) + f(x_5)\big)\Big] + \tfrac{\sqrt{\pi}}{36495360} f^{(12)}(\xi),$$

of degree of exactness eleven.

If we assume that $w(x) = e^{-x^2}$, $\omega(x) = x^4$, $m = 3$ and $(a,b) = (-\infty, \infty)$ we get the Gaussian generalized quadrature formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\mathrm{d}x = \tfrac{\sqrt{\pi}}{16464}\Big\{15744 f(0) + 2856 f''(0) + 147 f^{(IV)}(0)$$

$$+ 360\Big[f\big(-\tfrac{\sqrt{7}}{2}\big) + f\big(\tfrac{\sqrt{7}}{2}\big)\Big]\Big\} + \tfrac{\sqrt{\pi}}{552960} f^{(10)}(\xi)$$

of degree of exactness nine.

7. IMPORTANT FINAL REMARKS

In the papers [4], [5], [6] of W. Gautschi, there are called Gauss–Christoffel quadrature formulae the Gaussian quadrature formulae for weighted integrals. But we consider that the principal contribution of E. B. Christoffel [1] was to introduce in quadrature formulae some preassigned (fixed) nodes and to maximize the degree of exactness of such a quadrature formula.

Gaussian quadrature formulae for weighted integral were considered by G. C. Jacobi [7], F. G. Mehler [10], C. Posse [11], later by E. B. Christoffel (in a second paper [2]), T. J. Stieltjes [17], A. Markov [9], J. Deruyts [3] and others.

In the paper [15] D. D. Stancu has investigated an extended generalization of the P. Turan [19] quadrature formula by using multiple fixed nodes and multiple Gaussian nodes having odd orders of multiplicities.

We mention that in the paper [16] D. D. Stancu, in collaboration with A. H. Stroud, has tabulated the values of the Gaussian nodes, the coefficients and the remainders, with 20 significant digits, for several weighted quadrature formulae using different multiple fixed nodes.

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