REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 32 (2003) no. 2, pp. 235–242
ictp.acad.ro/jnaat

EFFICIENCY AND GENERALIZED CONCAVITY FOR MULTIOBJECTIVE SET-VALUED PROGRAMMING

Ş. ŢIGAN* and I. M. STANCU-MINASIAN†

Abstract. The purpose of this paper is to give sufficient conditions of generalized concavity type for a local (weakly) efficient solution to be a global (weakly) efficient solution for an vector maximization set-valued programming problem. In the particular case of the vector maximization set-valued fractional programming problem, we derive some characterizations properties of efficient and properly efficient solutions based on a parametric procedure associated to the fractional problem.

MSC 2000. 49J53, 90C29, 90C32.

Keywords. Vector set-valued programming, fractional programming, efficiency, proper efficiency, semistrict quasiconcavity, semiexplicit quasiconcavity.

1. INTRODUCTION

Let $X \subset \mathbb{R}^n$ and $Q : X \longrightarrow \mathcal{P}(\mathbb{R}^p)$ be a set-valued map defined on X, where $\mathcal{P}(\mathbb{R}^p)$ denotes the family of subsets in $\mathbb{R}^p$.

The vector maximization set-valued programming problem is formulated as

$$\text{(MSVP)} \qquad \text{Vmax}\, Q(x), \quad \text{subject to } x \in X.$$

The optimal solutions of the MSVP that we deal with include the concepts of efficient, weakly efficient, local efficient and properly efficient solutions that will be defined with respect to a semiorder relationship between subsets in $\mathbb{R}^p$.

The paper is organized as follows. In Section 2 we introduce the notations and definitions, which will be used throughout of the paper.

In Section 3, we give sufficient conditions of generalized concavity type for a local (weakly) efficient solution to be a global (weakly) efficient solution of an MSVP.

In Section 4, for the particular case of the vector maximization set-valued fractional programming problem, we obtain some characterizations properties of efficient and properly efficient solutions by using a parametric auxiliary problem.

Some concluding remarks are made in the last section.

*Department of Medical Informatics, University of Medicine and Pharmacy "Iuliu Haţieganu", Cluj-Napoca, Romania, e-mail: stigan@umfcluj.ro.

†Institute of Mathematical Statistics and Applied Mathematics, Romanian Academy, Bucharest, Romania, e-mail: stancum@csm.ro.

2. NOTATION AND DEFINITIONS

Let $A, B \subset \mathbb{R}^p$ be non-empty subsets in $\mathbb{R}^p$ and θ a given real number. Then we define the following set operations and relations:

(i) $A + B = \{a + b \mid a \in A,\ b \in B\}$;
(ii) $\mathrm{Min}(A, B) = \{\min(a, b) | a \in A, b \in B\}$;
(iii) $\theta\ A = \{\theta a \mid a \in A\}$;
(iv) $\frac{A}{B} = \{\frac{a}{b} = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_p}{b_p}) \mid a \in A, b \in B\}$, for any $B \subset \mathbb{R}^p_+$, where $\mathbb{R}^p_+$ denotes the positive orthant of $\mathbb{R}^p$;
(v) $A \geq B$ if for each $b \in B$, there exists $a \in A$ such that $a \geq b$, that is, $a_i \geq b_i$, for any $i \in J = \{1, 2, \dots, p\}$;
(vi) $A > B$ if for each $b \in B$, there exists $a \in A$ such that $a > b$, that is, $a_i \geq b_i$, for any $i \in J$ and there is $j \in J$ such that $a_j > b_j$;
(vii) $A \gg B$ if for each $b \in B$, there exists $a \in A$ such that $a \gg b$, that is, $a_i > b_i$, for any $i \in J$.

Next, we consider some classes of generalized concave functions.

Let $f : X \longrightarrow \mathcal{P}(\mathbb{R}^p)$ be a set-valued function defined on the convex non-empty set X in $\mathbb{R}^n$.

a) The function f is said to be *quasiconcave* if for any $x', x'' \in X$ and $t \in (0, 1)$,

$$f\big(tx' + (1 - t)x''\big) \geq \mathrm{Min}\,\big(f(x'), f(x'')\big).$$

b) The function f is said to be *semistrictly quasiconcave* if for any $x', x'' \in X$ such that $f(x') \neq f(x'')$, we have

$$f\big(tx' + (1 - t)x''\big) > \mathrm{Min}\,\big(f(x'), f(x'')\big), \quad \forall t \in (0, 1).$$

c) The function f is said to be *semiexplicitly quasiconcave* if it is quasiconcave and semistrictly quasiconcave.

d) The function f is said to be *strictly quasiconcave* if for all $x', x'' \in X$ such that $f(x') \neq f(x'')$, we have

$$f\big(tx' + (1 - t)x''\big) \gg \mathrm{Min}\,\big(f(x'), f(x'')\big), \quad \forall t \in (0, 1).$$

e) The function f is said to be *explicitly quasiconcave* if it is quasiconcave and strictly quasiconcave.

Obviously, f is semiexplicitly quasiconcave if it is explicitly quasiconcave. However, the converse is not true (see, [5]).

Next we consider for Problem MSVP some efficiency concepts based on the semiorder relationships presented above (see, (v)–(vii)) between the subsets in $\mathbb{R}^p$.

Definition 1. *A point $\bar{x} \in X$ is said to be an efficient solution to Problem* MSVP *if there does not exist $y \in X$ such that $Q(y) > Q(\bar{x})$.*

Let E denote the set of all efficient solutions to Problem MSVP.

Definition 2. *A point* $\bar{x} \in X$ *is said to be a weakly efficient solution to Problem* MSVP *if there does not exist* $y \in X$ *such that* $Q(y) \gg Q(\bar{x})$.

Let WE denote the set of all efficient solutions to Problem MSVP.

Definition 3. *A point* $\bar{x} \in X$ *is said to be a local* (*weakly*) *efficient solution to Problem* MSVP *if there does not exist* $y \in X \cap U$ *such that* $Q(y) > Q(\bar{x})$ $(Q(y) \gg Q(\bar{x}))$ *for some neighborhood* U *of* $\bar{x}$.

Let LE (LWE) denote the set of all local (weakly) efficient solutions to Problem MSVP.

Definition 4. *An efficient solution* $\bar{x} \in X$ *to Problem* MSVP *is said to be a properly efficient solution if there exists* $\bar{u} \in Q(\bar{x})$ *and a scalar* $M > 0$ *such that for all* $i \in J$ *and each* $x \in X$, *for which there exists* $u \in Q(x)$ *such that* $u_i > \bar{u}_i$, *there exists* $j \in J - \{i\}$, *for which* $u_j > \bar{u}_j$ *and* $\frac{u_i - \bar{u}_i}{\bar{u}_j - u_j} \leq M$.

Let PE denote the set of all properly efficient solutions to Problem MSVP.

Obviously, from Definitions 1–4 we have the following relationship between the different classes of optimal solutions of MSVP:

$$(1) \qquad PE \subset E \subset WE, \quad E \subset LE, \quad WE \subset LWE, \quad LE \subset LWE.$$

The efficiency notions given by Definitions 1–3 are analogous to that considered for vector real valued objective functions in refs. [1], [5], [6]. The proper efficiency concept is a generalization of that introduced by Geoffrion [3]. We also mention that in the case of vector real valued objective functions relationships between another types of proper efficiency solutions was studied by Giorgi and Guerraggio [4].

3. GENERALIZED CONCAVE CONDITIONS FOR LOCAL-GLOBAL PROPERTIES

We now generalize to set-valued vector optimization problems a characterization of local efficient solutions obtained in [5] and [6] for usual vector optimization problems. A similar result is given for local weakly efficient solutions.

Theorem 5. *Let* $Q : X \longrightarrow \mathcal{P}(\mathbb{R}^p)$ *be a set-valued semiexplicitly quasiconcave function, where* $X \subset \mathbb{R}^n$ *is a non-empty convex set. Then* $\bar{x} \in X$ *is a local efficient solution to Problem* MSVP *if and only if* $\bar{x}$ *is a* (*global*) *efficient solution* (*i.e.,* $E = LE$).

Proof. From (1) we have $E \subset LE$. To prove the converse inclusion, assume to the contrary that $\bar{x} \in X$ is a local efficient solution (with respect to a neighborhood U of $\bar{x}$), which is not a global efficient solution. Then there exists $y \in X$ such that $Q(y) > Q(\bar{x})$. Since Q is semiexplicitly quasiconcave, we have

$$(2) \qquad Q(ty + (1-t)\bar{x}) > Q(\bar{x}), \quad \text{for all } t \in (0, 1).$$

But for t sufficiently close to zero, $x(t) = ty + (1-t)\bar{x}$ will be in the neighborhood U of $\bar{x}$. But this shows by (2) and Definition 1 that $\bar{x}$ would not be a local efficient solution, which is a contradiction. □

We supplement the result in Theorem 5 by a similar one for weakly efficient solutions.

THEOREM 6. *Let* $Q : X \longrightarrow \mathcal{P}(\mathbb{R}^p)$ *be a set-valued explicitly quasiconcave function, where* $X \subset \mathbb{R}^n$ *is a non-empty convex set. Then* $\bar{x} \in X$ *is a local weakly efficient solution of* MSVP *if and only if* $\bar{x}$ *is a (global) weakly efficient solution.*

Proof. One can follow the lines of previous proof. To prove the nontrivial implication, assume to the contrary that $\bar{x} \in X$ is a local weakly efficient solution (with respect to a neighborhood U of $\bar{x}$), which is not a global weakly efficient solution. Then there exists $y \in X$ such that $Q(y) \gg Q(\bar{x})$. Since Q is explicitly quasiconcave, we have

$$Q(ty + (1-t)\bar{x}) \gg Q(\bar{x}), \quad \text{for all } t \in (0,1). \tag{3}$$

But for t sufficiently close to zero, $x(t) = ty + (1-t)\bar{x}$ will be in the neighborhood U of $\bar{x}$. But this shows by (3) and Definition 2 that $\bar{x}$ would not be a local weakly efficient solution, which is a contradiction. □

In contrast to what observed for efficient solutions in Theorem 5, the more general assumption of semiexplicit quasiconvexity is no longer sufficient to prove the local-global property for weakly efficient solutions. To see this, if in the previous proof the inequality (3) would be replaced by (2) (that holds in the case of semiexplicit quasiconvexity assumption) then the final conclusion is no longer true. Moreover, Luc and Schaible [5] give an example for vector real valued objective functions, which shows that semiexplicit quasiconvexity assumption is no longer sufficient to obtain the local-global property for weakly efficient solutions.

4. MULTI-OBJECTIVE FRACTIONAL SET-VALUED PROGRAMMING PROBLEM

The vector maximization fractional set-valued problem is formulated as

$$\text{(FSVP)} \qquad \text{Vmax}\, \frac{F(x)}{G(x)} \equiv \left(\frac{F_1(x)}{G_1(x)}, \ldots, \frac{F_p(x)}{G_p(x)} \right), \quad \text{subject to } x \in X,$$

where $F : X \longrightarrow \mathbb{R}^p$ and $G : X \longrightarrow \mathbb{R}^p$ are vector set-valued maps, and for each $x \in X$, and $i = 1, 2, \ldots, p$, we assume that

$$G_i(x) \subset \mathbb{R}^p_+. \tag{4}$$

Bhatia and Mehra [1] studied Lagrangian duality for vector optimization fractional set-valued problem FSVP.

Let

$$Q(x) = \frac{F(x)}{G(x)}, \quad \forall x \in X.$$

We mention that some particular cases of vector maximization fractional set-valued problems was studied via inexact programming in [7], [8] and [9].

In the fractional case, by using a parametric approach, we consider for a p-dimensional vector $\theta = (\theta_1, \dots, \theta_p) \in \mathbb{R}^p$, the following vector set-valued optimization problem

$$(\mathrm{SVP}(\theta)) \qquad \mathrm{Vmax}\, H(\theta, x) \equiv (F_1(x) - \theta_1 G_1(x), \dots, F_p(x) - \theta_p G_p(x)),$$

subject to $x \in X$.

The following theorem generalizes similar results obtained by Ţigan [10] and Chandra, Craven and Mond [2] for real valued objective functions and Bhatia and Mehra [1] for set-valued objective functions.

Theorem 7. *Let $x' \in X$. Then x' is efficient* (*weakly efficient*) *for problem $FSVP$ with $u' = \frac{y'}{z'} \in Q(x')$, $y' \in F(x')$, $z' \in G(x')$ as an efficient* (*a weakly efficient*) *value of* FSVP *if and only if x' is efficient* (*weakly efficient*) *for problem* SVP(u'), *where* $0 \in F(x) - u'G(x)$ *as an efficient* (*weakly efficient*) *value of* SVP(u').

Proof. i) Assume that x' is an efficient solution for problem FSVP. Let $x \in X$ and $i \in J$ for which there exists $u = y/z \in Q(x)$ such that $u_i > u'_i$. Then, since x' is efficient for FSVP, there exists $j \in J \setminus \{i\}$ such that $u_j < u'_j$. By assumption (4), the inequality $u_j = \frac{y_j}{z_j} < u'_j$ is equivalent to $v_j = y_j - u'_j z_j < 0 = v'_j = y'_j - u'_j z'_j \in H(u', x')$.

Therefore, for every $x \in X$ and $i \in J$ for which $v_i = y_i - u'_i z_i > 0 = y'_i - u'_i z'_i = v'_i$, there exists $j \in J \setminus \{i\}$ such that $v_j = y_j - u'_j z_j < 0 = v'_j = y'_j - u'_j z'_j$. Hence, it follows that x' is an efficient solution for SVP(u').

ii) The proof of converse part of the theorem follows the same lines as the direct part.

The proof for weakly efficient solutions is analogous. □

In order to obtain a similar result for the properly efficient solutions of FSVP, we need the following assumption:

There exists $M_1 > 0$, such that

$$(5) \qquad \sup\left\{\tfrac{y_i}{y_j} \mid y_i \in G_i(x), y_j \in G_j(x), i \in J, j \in J \setminus \{i\}\right\} \le M_1, \quad \forall x \in X.$$

The following theorem generalizes a similar result obtained by Ţigan [10].

Theorem 8. *Let $u' = \frac{y'}{z'} \in Q(x')$, $y' \in F(x')$, $z' \in G(x')$ be an efficient value of* FSVP, *where $y' \in F(x')$, $z' \in G(x')$. If the assumptions* (4) *and* (5) *hold, then $x' \in X$ is properly efficient solution for* FSVP *if and only if it is properly efficient for* SVP(u').

Proof. i) Let x' be a properly efficient solution of FSVP. We must show that x' is a properly efficient solution of SVP(u'), where $u' = y'/z' \in Q(x')$. Let $i \in J$ and $x \in X$ and $u_i = y_i/z_i \in Q_i(x)$, $y_i \in F_i(x)$, $z_i \in G_i(x)$ such that:

$$(6) \qquad y_i - u'_i z_i > 0 = y'_i - u'_i z'_i,$$

where $y_i - u'_i z_i \in H_i(u', x) = F_i(x) - u'_i G_i(x)$ and $0 = y'_i - u'_i z'_i \in H_i(u', x') = F_i(x') - u'_i G_i(x')$.

Then, from (4), (5) and (6), since $z_i > 0$, it follows that

$$\frac{y_i}{z_i} > u' = \frac{y'_i}{z'_i}. \tag{7}$$

Since x' is properly efficient for FSVP, it results from (7) that there exists $M > 0$ and $j \in J \setminus \{i\}$ such that $\frac{y_j}{z_j} < \frac{y'_j}{z'_j}$, where $y_j \in F_j(x)$, $z_j \in G_j(x)$, $y'_j \in F_j(x')$, $z_j \in G_j(x')$ and

$$\frac{\frac{y_i}{z_i} - \frac{y'_i}{z'_i}}{\frac{y'_j}{z'_j} - \frac{y_j}{z_j}} \le M. \tag{8}$$

Then, from (4), (5) and (8), we have

$$\frac{y_i - u'_i z_i}{z_j u'_j - y_j} = \frac{\frac{y_i}{z_i} - \frac{y'_i}{z'_i}}{\frac{y'_j}{z'_j} - \frac{y_j}{z_j}} \cdot \frac{z_i}{z_j} \le M \cdot M_1. \tag{9}$$

Therefore, it follows that x' is properly efficient for SVP(u').

ii) Now let suppose that $x' \in X$ is a properly efficient solution for SVP(u') and, furthermore, for a certain $x \in X$ and $i \in J$, there exists $u = y/z \in Q(x)$ such that $u_i > u'_i$. Then, by (4), from $u_i > u'_i$ it follows that (6) holds. Since x' is a properly efficient solution for SVP(u'), there exists $j \in J \setminus \{i\}$ and $M'' > 0$ such that the inequality

$$v_j = y_j - u'_j z_j < 0 = v'_j = y'_j - u'_j z'_j \in H(u', x') \tag{10}$$

holds and

$$\frac{y_i - u'_i z_i}{z_j u'_j - y_j} \le M''. \tag{11}$$

But by (4) the inequality (9) is equivalent to

$$\frac{y_j}{z_j} < \frac{y'_j}{z'_j},$$

and from (5) and (10) it results that (8) is verified with $M = M'' \times M_1$. This means that x' is properly efficient for FSVP.

5. CONCLUSIONS

In this paper we obtained sufficient conditions implying generalized concavity assumptions of the set-valued functions, in order to a local (weakly) efficient solution be a global (weakly) efficient solution for an vector maximization set-valued programming problem.

In the particular case of the vector maximization set-valued fractional programming problem, we derived some characterizations properties of efficient and properly efficient solutions via a parametric procedure associated to the fractional problem.

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Received by the editors: March 12, 2002.