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ON COMPOUND OPERATORS CONSTRUCTED WITH BINOMIAL AND SHEFFER SEQUENCES*

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Abstract. In this note we consider a general compound approximation operator using binomial sequences and we give a representation for its corresponding remainder term. We also introduce a more general compound approximation operator using Sheffer sequences. We provide convergence theorems for both studied operators.

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1. INTRODUCTION

A sequence of polynomials $(p_m(x))_{m\geq 0}$ is called a sequence of binomial type if $\deg p_m = m$, $\forall m \in \mathbb{N}$, and satisfies the identities

$$p_m(x+y) = \sum_{k=0}^{m} \tbinom{m}{k} p_k(x)\, p_{m-k}(1-x)\,.$$

We will denote by E^a the shift operator defined by $(E^a p)(x) = p(x+a)$, for every polynomial p and every real number x.

A linear operator T is said to be shift invariant if it commutes with the shift operator E^a, for every real number a.

Sequences of binomial type are connected with the notion of theta operators (J. F. Steffensen [27], [28]) which were called delta operators by F. B. Hildebrand [7] and G.-C. Rota and his collaborators [15].

A delta operator Q is a shift invariant operator for which $Qx = \text{const.} \neq 0$.

Definition 1. *Let Q be a delta operator. A sequence $(p_m(x))_{m\geq 0}$ is a sequence of basic polynomials for Q (basic sequence, for short) if:*

i) $p_0 = 1$,
ii) $p_m(0) = 0$, *if* $m \geq 1$,
iii) $Qp_m = mp_{m-1}$, *if* $m \geq 1$.

For every delta operator there exists a unique basic sequence. A polynomial sequence is a sequence of binomial type if and only if it is the sequence of basic polynomials for a delta operator.

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Sequences of binomial type were called poweroids by Steffensen [27], because the action of any delta operator on the binomial sequence, which is its basic sequence, is the same as the action of the derivative D on x^m.

Definition 2. *A sequence of polynomials* $(s_m(x))_{m\geq 0}$ *is called a Sheffer sequence for* Q *if:*

i) $s_0 = \text{const.} \neq 0$,

ii) $Qs_m = ms_{m-1}$, *if* $m \geq 1$.

It is known [15] that if $(s_m(x))_{m\geq 0}$ is a Sheffer sequence for a delta operator Q with the basic sequence $(p_m(x))_{m\geq 0}$ then there exists a shift invariant and invertible operator S such that $s_m = S^{-1}p_m$, $\forall m \in \mathbb{N}$, so every pair (Q, S) gives us a unique Sheffer sequence.

A Sheffer sequence satisfies the relations

$$s_m(x+y) = \sum_{k=0}^{m} \binom{m}{k} p_k(x)\, s_{m-k}(1-x), \quad \forall m \in \mathbb{N}. \tag{1}$$

A Sheffer sequence for the usual derivative D is an Appell sequence.

The Umbral Calculus allows a unified and simple study of binomial, Appell and Sheffer sequences. More details about Umbral Calculus can be found in [15], [5] and [6].

T. Popoviciu proposed in [14] the use of binomial sequences in order to construct a class of approximation operators of the form

$$\left(T_m^Q f\right)(x) = \frac{1}{p_m(1)} \sum_{k=0}^{m} \binom{m}{k} p_k(x)\, p_{m-k}(1-x)\, f\left(\frac{k}{m}\right), \tag{2}$$

for every function $f \in C[0,1]$.

This kind of operators and their generalizations were intensively studied. They interpolate the function f at 0 and 1 and preserve the polynomials of degree one. The expressions for $T_m^Q e_n$, $n \geq 2$, were computed by C. Manole (see [10] and [11]) using the umbral calculus and later by P. Sablonnière using the generating function for the binomial sequences (see [16]). We mention that Sablonnière called them Bernstein–Sheffer operators while D. D. Stancu called them binomial operators of Tiberiu Popoviciu type.

Different results regarding the operator T_m^Q were obtained by several authors: D. D. Stancu and M. R. Occorsio found representations for the remainder in the approximation formula $f(x) = (T_m^Q f)(x) + (R_m^Q f)(x)$ [24]; V. Miheşan proved that T_m^Q preserve the Lipschitz constant for a Lipschitz function [12]; D. D. Stancu and A. Vernescu studied bivariate operators of this type [26]; O. Agratini considered a generalization of T_m^Q in the Kantorovich sense [1]; L. Lupaş and A. Lupaş introduced and studied a modified operator of binomial type replacing x by mx and 1 by m [9], [8]. More details about the role of the binomial polynomials in the Approximation Theory can be found in [2], [8] and [24].

2. COMPOUND POWEROID OPERATORS

Let Q be a delta operator with the basic sequence $(p_m(x))_{m\geq 0}$. If $p_m(1)\neq 0$, $\forall m\in\mathbb{N}$, for every function $f\in C[0,1]$ we consider the general approximation operator defined by

$$\left(S^Q_{m,r,s}f\right)(x)=\sum_{k=0}^{m-sr}p^Q_{m-sr,k}(x)\sum_{j=0}^{s}p^Q_{s,j}(x)\,f\left(\tfrac{k+jr}{m}\right), \tag{3}$$

where $p^Q_{n,k}(x)=\binom{n}{k}\frac{p_k(x)p_{n-k}(1-x)}{p_n(1)}$, while s and r are two nonnegative integers satisfying the condition $2sr\leq m$.

If $p'_m(0)\geq 0$, $\forall m\in\mathbb{N}$, then $p_m(x)\geq 0$, $\forall x\in[0,1]$, so the operator $S^Q_{m,r,s}f$ is a positive approximation operator.

Different instances of this compound poweroid operator were previously studied by D. D. Stancu and his collaborators as follows:

1. For $Q=D$, $p_k(x)=x^k$, $s=1$ the corresponding compound operator was introduced and studied by D. D. Stancu (see [18]); if s is arbitrary, the operator $S^D_{m,r,s}$ is a special case of the operator $L^{\alpha,\beta}_{m,r_1,\ldots,r_s}$, considered by D. D. Stancu in [19] (in fact $S^D_{m,r,s}$ is obtained from $L^{\alpha,\beta}_{m,r_1,\ldots,r_s}$ when $\alpha=\beta=0$ and $r_1=r_2=\ldots=r_s=r$);
2. The case obtained for $Q=\frac{1}{\alpha}\nabla_\alpha=\frac{I-E^{-\alpha}}{\alpha}$, $p^\alpha_k(x)=x^{[k,-\alpha]}$ was studied by D. D. Stancu and J. W. Drane in [23];
3. D. D. Stancu and A. C. Simoncelli studied in [25] the compound poweroid operator for $Q=\frac{1}{\alpha}E^{-\beta}\nabla_\alpha=\frac{1}{\alpha}\left(E^{-\beta}-E^{-\alpha-\beta}\right)$, $p^{\alpha,\beta}_k(x)=x(x+\alpha+k\beta)^{[k-1,-\alpha]}$. They proved that if $\alpha=\alpha(m)\to 0$, $m\beta(m)\to 0$, as $m\to\infty$, then $(S^{\alpha,\beta}_{m,r,s}f)$ converges uniformly to f on the interval $[0,1]$. Using the Peano theorem, the authors also gave a representation of the remainder $R^{\alpha,\beta}_{m,r,s}f$ for the approximation formula

$$f(x)=(S^{\alpha,\beta}_{m,r,s}f)(x)+(R^{\alpha,\beta}_{m,r,s}f)(x)$$

D. D. Stancu considered also a class of linear positive compound operators $S^{\alpha,\beta,\gamma,\delta}_{m,r,s}f$ (see [22]) with modified knots defined by the following relation

$$(S^{\alpha,\beta,\gamma,\delta}_{m,r,s}f)(x)=\sum_{k=0}^{m-sr}p^{\alpha,\beta}_{m-sr,k}(x)\sum_{j=0}^{s}p^{\alpha,\beta}_{s,j}(x)f\left(\tfrac{k+jr+\gamma}{m+\delta}\right),$$

where $0\leq\gamma\leq\delta$.

If $s=0$ or $r=0$ then $S^Q_{m,r,0}$ and $S^Q_{m,0,s}$ reduce to the binomial operator of T. Popoviciu T^Q_m, defined by (2).

From the definition of a basic sequence it results that

$$p^Q_{n,k}(0)=\begin{cases}1, & \text{if } k=0\\ 0, & \text{if } k\neq 0\end{cases}\qquad\text{and}\quad p^Q_{n,k}(1)=\begin{cases}1, & \text{if } k=n\\ 0, & \text{if } k\neq n.\end{cases}$$

Using these relations we obtain that the polynomial $S^Q_{m,r,s}f$ interpolates f at both sides of the interval $[0,1]$, that is, $(S^Q_{m,r,s}f)(0) = f(0)$, $(S^Q_{m,r,s}f)(1) = f(1)$.

LEMMA 3. *The values of the operator* $S^Q_{m,r,s}$ *for the test functions are*

$$\begin{aligned} \left(S^Q_{m,r,s}e_0\right)(x) &= e_0(x), \\ \left(S^Q_{m,r,s}e_1\right)(x) &= e_1(x), \\ \left(S^Q_{m,r,s}e_2\right)(x) &= x^2 + x(1-x)A^Q_{m,s,r}, \end{aligned} \tag{4}$$

where $A^Q_{m,s,r} = \frac{(m-sr)^2 d^Q_{m-sr} + r^2 s^2 d^Q_s}{m^2}$ *and* $d^Q_m = 1 - \frac{m-1}{m}\frac{(Q')^{-2}p_{m-2}(1)}{p_m(1)}$.

Proof. From the definition of a sequence of binomial type we have that $\sum_{k=0}^m p^Q_{m,k}(x) = 1$, so it follows that

$$\begin{aligned} \left(S^Q_{m,r,s}e_0\right)(x) &= \sum_{k=0}^{m-sr} p^Q_{m-sr,k}(x) \sum_{j=0}^{s} p^Q_{s,j}(x) = 1 = e_0(x), \\ \left(S^Q_{m,r,s}e_1\right)(x) &= \tfrac{1}{m}\sum_{k=0}^{m-sr} p^Q_{m-sr,k}(x)\Big[k\sum_{j=0}^{s} p^Q_{s,j}(x) + r\sum_{j=0}^{s} jp^Q_{s,j}(x)\Big] \\ &= \tfrac{1}{m}\sum_{k=0}^{m-sr} p^Q_{m-sr,k}(x)\Big[k\left(T^Q_s e_0\right)(x) + rs\left(T^Q_s e_1\right)(x)\Big] \\ &= \tfrac{1}{m}\left[(m-sr)(T^Q_{m-sr}e_1)(x) + xrs(T^Q_{m-sr}e_0)(x)\right] \\ &= \tfrac{1}{m}\left[(m-sr)x + rsx\right] \\ &= x. \end{aligned}$$

Hence the operator $S^Q_{m,r,s}$ preserves the polynomials of degree one.

Analogously, we obtain that

$$\begin{aligned} &(S^Q_{m,r,s}e_2)(x) = \\ &= \tfrac{1}{m^2}\Big[(m-sr)^2(T^Q_{m-sr}e_2)(x) + 2(m-sr)srx^2 + r^2s^2(T^Q_s e_2)(x)\Big]. \end{aligned}$$

Using the expression found by C. Manole in [11] for $T^Q_m e_2$,

$$\left(T^Q_m e_2\right)(x) = x^2 + x(1-x)d^Q_m,$$

with $d^Q_m = 1 - \frac{m-1}{m}\frac{(Q')^{-2}p_{m-2}(1)}{p_m(1)}$, we obtain

$$\left(S^Q_{m,r,s}e_2\right)(x) = x^2 + x(1-x)A^Q_{m,s,r},$$

where $A^Q_{m,s,r} = \frac{(m-sr)^2 d^Q_{m-sr} + r^2 s^2 d^Q_s}{m^2}$. □

If $Q = D$ then $d_m^D = \frac{1}{m}$ and $A_{m,s,r}^D = \frac{m+sr(r-1)}{m^2}$.

For $Q = \frac{\nabla_\alpha}{\alpha}$ we have $d_m^{\frac{\nabla_\alpha}{\alpha}} = \frac{1+\alpha m}{(1+\alpha)m}$ so it follows that

$$A_{m,s,r}^{\frac{\nabla_\alpha}{\alpha}} = \frac{sr^2(1+\alpha s)+(m-sr)(1+\alpha(m-sr))}{(1+\alpha)m^2}.$$

From the relations (4) it results that $(S_{m,r,s}^Q e_2)(x)$ converges to $e_2(x)$ if $A_{m,s,r}^Q \to 0$, as $m \to \infty$. But, if $d_m^Q \to 0$, then $A_{m,s,r}^Q \to 0$, so using the Bohman–Korokvin criterion of convergence we have the following result.

Theorem 4. *Let Q be a delta operator with the basic sequence $(p_m(x))_{m\geq 0}$, $p_m(1) \neq 0$ and $p_m'(0) \geq 0$, for every positive integer m. If $d_m^Q \to 0$ then the sequence of linear and positive operators $S_{m,r,s}^Q f$ converges to the function f, uniformly on the interval* $[0,1]$.

Now we establish an estimate for the order of approximation of a function $f \in C[0,1]$ by means of the operator $S_{m,r,s}^Q$ using the first modulus of continuity.

Taking into account an inequality proved by O. Shisha and B. Mond (see [17]), we can write

$$\left|f(x) - (S_{m,r,s}^Q f)(x)\right| \leq \left[1 + \frac{1}{\delta^2} S_{m,r,s}^Q((t-x)^2;x)\right]\omega_1(f;\delta).$$

Using the expressions obtained for $S_{m,r,s}^Q e_i$, for $i = 0,1,2$, we obtain that $S_{m,r,s}^Q((t-x)^2;x) = x(1-x)A_{m,s,r}^Q$. Taking into account that $x(1-x) \leq \frac{1}{4}$, $\forall x \in [0,1]$ and replacing δ by $\sqrt{A_{m,s,r}^Q}$, we obtain that

$$\left|f(x) - (S_{m,r,s}^Q f)(x)\right| \leq \tfrac{5}{4}\omega_1\left(f; \sqrt{A_{m,s,r}^Q}\right).$$

Example 1. If we consider the delta operator $T = \ln(I + D)$, its basic sequence is the sequence of exponential polynomials:

$$\varphi_m(x) = \sum_{k=1}^{m} S(m,k)\, x^k,$$

where $S(m,k) = [0,1,\ldots,k;e_m]$ are the Stirling numbers of second kind. In this case, Manole obtained

$$d_m^T = \frac{1}{m} + \frac{m-1}{m}\frac{\varphi_{m-1}(1)}{\varphi_m(1)},$$

and he proved that there exist two positive constants c_1 and c_2 such that

$$c_1 \frac{\ln m}{m} \leq \frac{\varphi_{m-1}(1)}{\varphi_m(1)} \leq c_2 \frac{\ln m}{m}.$$

Hence, $\frac{\varphi_{m-1}(1)}{\varphi_m(1)} \to 0$, as $m \to \infty$, which implies $d_m^T \to 0$. Consequently, $S_{m,r,s}^T f$ defined by

$$S_{m,r,s}^T = \sum_{k=0}^{m-sr} \frac{\varphi_k(x)\varphi_{m-sr-k}(1-x)}{\varphi_{m-sr}(1)} \sum_{j=0}^{s} \frac{\varphi_j(x)\varphi_{s-j}(1-x)}{\varphi_s(1)} f\left(\frac{k+jr}{m}\right)$$

converges to the function f, uniformly on the interval $[0,1]$. □

3. EVALUATION OF THE REMAINDER

Using Peano's theorem, the remainder of the approximation formula

$$f(x) = \left(S^{Q}_{m,r,s} f\right)(x) + \left(R^{Q}_{m,r,s} f\right)(x) \tag{5}$$

for $f \in C^2[0,1]$ can be represented as

$$\left(R^{Q}_{m,r,s} f\right)(x) = \int_0^1 G^{Q}_{m,r,s}(t;x) f''(t)\,\mathrm{d}t, \tag{6}$$

where $G^{Q}_{m,r,s}(t;x)$ is the Peano kernel defined by

$$G^{Q}_{m,r,s}(t;x) = \left(R^{Q}_{m,r,s}\varphi_x\right)(t), \qquad \varphi_x(t) = (x-t)_+ = \tfrac{1}{2}\left[x - t + |x-t|\right].$$

Since the expression

$$G^{Q}_{m,r,s}(t;x) = (x-t)_+ - \sum_{k=0}^{m-sr} p^{Q}_{m-sr,k}(x) \sum_{j=0}^{s} p^{Q}_{s,j}(x) \left(\tfrac{k+jr}{m} - t\right)_+$$

is negative, one can apply the mean value theorem to the integral from (6) and we obtain that there exists $\xi \in [0,1]$ such that

$$\left(R^{Q}_{m,r,s} f\right)(x) = f''(\xi) \int_0^1 G^{Q}_{m,r,s}(t;x)\,\mathrm{d}t.$$

Taking $f(x) = x^2$ in the previous relation, we obtain that

$$\int_0^1 G^{Q}_{m,r,s}(t;x)\,\mathrm{d}t = \tfrac{1}{2}\left(R^{Q}_{m,r,s} e_2\right)(x) = -\tfrac{1}{2} x(1-x) \frac{(m-sr)^2 d^{Q}_{m-sr} + r^2 s^2 d^{Q}_{s}}{m^2},$$

so it follows that, for every function $f \in C^2[0,1]$, the remainder in formula (5) is of the following form:

$$\left(R^{Q}_{m,r,s} f\right)(x) = \tfrac{x(x-1)}{2m^2}\left[(m-sr)^2 d^{Q}_{m-sr} + r^2 s^2 d^{Q}_{s}\right] f''(\xi).$$

4. COMPOUND SHEFFER OPERATORS

Let Q be a delta operator with the basic sequence $(p_m(x))$, S a shift invariant and invertible operator and $s_m = S^{-1}p_m$ a Sheffer sequence. We can also generalize the operator defined in (3), by considering another compound approximation operator containing a Sheffer sequence (additionally to the basic sequence p_m):

$$\left(S^{Q,S}_{m,r,s} f\right)(x) = \sum_{k=0}^{m-sr} w^{Q,S}_{m-sr,k}(x) \sum_{j=0}^{s} w^{Q,S}_{s,j}(x)\, f\left(\tfrac{k+jr}{m}\right), \tag{7}$$

where $w^{Q,S}_{n,k}(x) = \binom{n}{k} \frac{p_k(x) s_{n-k}(1-x)}{s_n(1)}$.

When $S = I$ this operator reduces to the operator defined by (3).

For $s=0$, $S^{Q,S}_{m,r,0}$ is in fact the operator that we studied in our paper [3], $\left(L^{Q,S}_{m}f\right)(x)=\sum_{k=0}^{m}w^{Q,S}_{m,k}(x)\,f\left(\frac{k}{m}\right)$. We remind that for this operator we have obtained the following expressions for the test functions

$$
\begin{aligned}
L^{Q,S}_{m}e_0 &= e_0\\
\left(L^{Q,S}_{m}e_1\right)(x) &= a_m e_1(x) \qquad (8)\\
\left(L^{Q,S}_{m}e_2\right)(x) &= b_m x^2 + x\left(a_m - b_m - c_m\right),
\end{aligned}
$$

where

$$a_m=\frac{(Q')^{-1}s_{m-1}(1)}{s_m(1)},\quad b_m=\frac{m-1}{m}\frac{\left[(Q')^{-2}s_{m-2}\right](1)}{s_m(1)},\quad c_m=\frac{m-1}{m}\frac{\left[(Q')^{-2}\left(S^{-1}\right)'p_{m-2}\right]}{s_m(1)}.$$

If $p'_n(0)\geq 0$ and $s_n(0)\geq 0$, $\forall n\in\mathbb{N}$, then $w^{Q,S}_{n,k}(x)\geq 0$, $\forall x\in[0,1]$ (see [3]) and so the operator $S^{Q,S}_{m,r,s}f$ is a positive approximation operator.

For the operator (7) we have $(S^{Q,S}_{m,r,s}f)(0)=f(0)$.

In the following we compute the values of the operator $S^{Q,S}_{m,r,s}$ for the test functions.

From the convolution-type relation (1) satisfied by a Sheffer sequence, it is obvious that

$$S^{Q,S}_{m,r,s}e_0=e_0.$$

For e_1 we have

$$
\begin{aligned}
\left(S^{Q,S}_{m,r,s}e_1\right)(x) &= \sum_{k=0}^{m-sr} w^{Q,S}_{m-sr,k}(x)\Big[\frac{k}{m}\left(L^{Q,S}_{s}e_0\right)(x)+\frac{rs}{m}\left(L^{Q,S}_{s}e_1\right)(x)\Big]\\
&= \frac{m-sr}{m}(L^{Q,S}_{m-sr}e_1)(x)+\frac{a_s rsx}{m}(L^{Q,S}_{m-sr}e_0)(x),
\end{aligned}
$$

and using the relations (8) we obtain that

$$\left(S^{Q,S}_{m,r,s}e_1\right)(x)=x\frac{(m-sr)a_{m-sr}+rsa_s}{m}.$$

Finally, for e_2 we have

$$
\begin{aligned}
\left(S^{Q,S}_{m,r,s}e_2\right)(x) = \frac{1}{m^2}\Big[&(m-sr)^2(L^{Q,S}_{m-sr}e_2)(x)\\
&+2rs(m-sr)(L^{Q,S}_{m-sr}e_1)(x)(L^{Q,S}_{s}e_1)(x)\\
&+r^2s^2\left(L^{Q,S}_{s}e_2\right)(x)\Big].
\end{aligned}
$$

Using again the relations (8) we can rewrite the last expression as

$$
\begin{aligned}
&\left(S^{Q,S}_{m,r,s}e_2\right)(x)=\\
&=\frac{1}{m^2}\Big\{x^2\left[(m-sr)^2 b_{m-sr}+2sr(m-sr)a_s a_{m-sr}+s^2r^2 b_s\right]\\
&\qquad +x\left[(m-sr)^2\left(a_{m-sr}-b_{m-sr}-c_{m-sr}\right)+s^2r^2(a_s-b_s-c_s)\right]\Big\}
\end{aligned}
$$

In [3] we proved that if $L_m^{Q,S}$ is a positive operator then

$$0 \le c_m \le \min\left\{\tfrac{1-b_m}{2}, a_m - a_m^2\right\}.$$

So, if $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = 1$, then $\lim_{m\to\infty} c_m = 0$. Taking into account the previous relations and applying the Bohman–Korokvin criterion convergence we can state the following result.

Theorem 5. *If* $f \in C[0,1]$ *and* $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = 1$, *then the sequence of compound operators constructed with Sheffer sequences defined by* (7) *converges uniformly to* f *on the interval* $[0,1]$.

Examples. 1. If we consider the special case when $Q = D$, then in the expression of $S_{m,r,s}^{D,S} f$, instead of s_m, there appears an Appell sequence A_m. Because the Pincherle derivative of D is the identity operator I, we have

$$a_m = \tfrac{A_{m-1}(1)}{A_m(1)}, \quad b_m = \tfrac{m-1}{m} a_m a_{m-1}, \quad c_m = \tfrac{m-1}{m} a_m \left(1 - a_{m-1}\right),$$

so the condition for the convergence of the operator $S_{m,r,s}^{D,S}$ is $\lim\limits_{m\to\infty} \frac{A_{m-1}(1)}{A_m(1)} = 1$.

2. If we take the Gould delta operator

$$G = \tfrac{1}{\alpha} E^{-\beta} \nabla_\alpha = \tfrac{1}{\alpha}\left(E^{-\beta} - E^{-\alpha-\beta}\right)$$

and the invertible operator

$$S = E^{\alpha+\beta} G' = \tfrac{1}{\alpha}\left(\left(\alpha+\beta\right) I - \beta E^{\alpha}\right)$$

then the corresponding basic sequence and Sheffer sequence are

$$p_m(x) = x\left(x+\alpha+m\beta\right)^{[m-1,-\alpha]}, \quad \text{resp.} \quad s_m(x) = \left(x+m\beta\right)^{[m,-\alpha]},$$

so in this case

$$w_{m,k}^{G,S}(x) = \tfrac{1}{(1+m\beta)^{[m,-\alpha]}} \tbinom{m}{k} x\left(x+\alpha+k\beta\right)^{[k-1,-\alpha]} \left(1-x+\left(m-k\right)\beta\right)^{[m-k,-\alpha]}.$$

We mention that the operator

$$(L_m^{G,S} f)(x) = \sum_{k=0}^{m} w_{m,k}^{G,S}(x) f\left(\tfrac{k}{m}\right)$$

was studied by G. Moldovan (see for example [13]).

If $\alpha \to 0$, $m\beta \to 0$, $\frac{m\beta}{\alpha} \to 0$, as $m \to \infty$, or $m\alpha \to 0$, $m\beta \to 0$, $\frac{m\beta}{\alpha} \to c$, as $m \to \infty$, then the sequence of operators $L_m^{G,S}$ converges uniformly to f on $[0,1]$.

It can be easily proved that, in the same conditions, the operator $S_{m,r,s}^{G,S}$ converges also uniformly to f on $[0,1]$. □

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