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ON SPLINE APPROXIMATION FOR BIVARIATE FUNCTIONS OF INCREASING CONVEX TYPE

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Abstract. The motivation of the paper is to construct the largest and smallest families of functions that allow us to generate the bivariate continuous stochastic orderings of increasing convex type introduced recently in Denuit et al. (1999). The main step will consist in deriving a spline approximation for bivariate continuous increasing convex functions, which extends to the bivariate case a fundamental result obtained by Popoviciu (1941).

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1. INTRODUCTION

The remarkable works of Tiberiu Popoviciu on the theory of convexity have deeply influenced certain research areas in numerical analysis, in theory of approximation and in functional analysis. A good idea of such developments can be found in the special issues 1–2 (vol. 26, 1997), of the *Revue d'Analyse Numérique et de Théorie de l'Approximation*, dedicated to the memory of Tiberiu Popoviciu.

Recently, another central role of the theory of convexity has also been pointed out in probability and statistics within the theory of stochastic orderings. This is not really surprising since the question of comparison is often encountered in the works of Tiberiu Popoviciu (as underlined, e.g., by E. Popoviciu [11]).

So, wide classes of stochastic orderings, univariate or bivariate, discrete or continuous, of (increasing) convex type have been introduced in order to compare random variables, univariate or bivariate, discrete or continuous. Roughly, a random variable is said to be smaller than another one in that sense that if the expectation of any (increasing) convex function of this random variable is smaller than for the other variable.

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A problem of interest on its own and for certain applications is the derivation of the largest and smallest families of functions that allow us to generate the orderings. This question has been solved in Denuit et al. [4] for the univariate continuous case. Our purpose in the present paper is to determine the extremal generators for the bivariate continuous increasing convex stochastic orderings. The main step here will consist in constructing a spline approximation for bivariate continuous increasing convex functions. This extends to the bivariate case a fundamental result obtained by Popoviciu [14] and reexamined later by Bojanic and Roulier [2] and Dadu [3], inter alia.

For the notation in the sequel, the real line is denoted by $\mathbb{R}$, the set of the non-negative integers by $\mathbb{N}$ and any point (x_1, x_2) of the real space $\mathbb{R}^2$ by an underlined small letter $\underline{x}$. The vector of ones, that is $(1,1)$, is written as $\underline{1}$; similarly, $\underline{2} = (2,2)$ and so on; $\underline{x} \pm \underline{y}$ stands for $(x_1 \pm y_1, x_2 \pm y_2)$. The space $\mathbb{R}^2$ is endowed with the usual componentwise partial order, that is $\underline{x} \leq \underline{y}$ if $x_i \leq y_i$ for $i = 1, 2$. Given $x \in \mathbb{R}$ and $k \in \mathbb{N}$, x_+^k is equal to x^k when $x^k > 0$ and 0 otherwise (with the convention that x_+^0 is equal to 1 when $x > 0$ and 0 otherwise).

2. BIVARIATE REAL FUNCTIONS AND STOCHASTIC ORDERINGS OF INCREASING CONVEX TYPE

A stochastic ordering is any binary relation defined on a set of probability measures and that allows us to compare any pair of these probability measures. Thus, it translates the notions of being greater or being more variable, for instance, to probability measures. Usually, the stochastic orderings under interest are partial orderings, i.e. binary relations $\preceq$ satisfying the properties of reflexivity, transitivity and anti-symmetry.

In practice, it is often more convenient to work with random variables rather than with probability measures. A random variable is said to be smaller than another random variable in the $\preceq$ sense when this ordering holds for their probability measures. Note that, as a consequence, the property of anti-symmetry is lost.

A number of stochastic orderings have been introduced during the last two decades, mostly motivated by different areas of applications (statistics, queueing theory, reliability theory, economics, biomathematics, actuarial sciences, physics...). They gave rise to a rich and abundant literature; see, e.g., the books by Shaked and Shanthikumar [15] and Stoyan [16], and the classified bibliography by Mosler and Scarsini [9].

A rather general class of bivariate stochastic orderings is the class of integral orderings generated by some cones of bivariate functions (see, e.g., Marshall [8]). Let $\underline{X}$ and $\underline{Y}$ be a pair of bivariate random variables assumed to be continuous and valued in an interval $[\underline{a}, \underline{b}] = [a_1, b_1] \times [a_2, b_2]$ of $\mathbb{R}^2$. Consider a cone $\mathcal{F}$ of measurable functions $\phi : [\underline{a}, \underline{b}] \to \mathbb{R}^2$. Then, $\underline{X}$ is said to be smaller than $\underline{Y}$ for the integral stochastic ordering $\preceq_{\mathcal{F}}^{[\underline{a},\underline{b}]}$ generated by $\mathcal{F}$ when

$$E\phi(\underline{X}) \le E\phi(\underline{Y}), \qquad \text{for all functions } \phi \in \mathcal{F}, \tag{1}$$

provided that the expectations exist.

In the present work, we will be concerned with a particular class of bivariate continuous integral orderings introduced in Denuit et al. [5]. This class is generated by the cone of the bivariate continuous increasing convex functions on $[\underline{a},\underline{b}]$, hence its appellation of bivariate continuous of increasing convex type.

Specifically, given any $\underline{s} \in \mathbb{N}^2$ with $s_1 + s_2 \ge 1$, let $\mathcal{U}_{\underline{s}-cx}^{[\underline{a},\underline{b}]}$ be the family of functions ϕ defined as

$$\mathcal{U}_{\underline{s}-cx}^{[\underline{a},\underline{b}]} = \Big\{\phi : [\underline{a},\underline{b}] \to \mathbb{R} \mid \phi^{(s_1,s_2)} \ge 0 \text{ on } [\underline{a},\underline{b}]\Big\}, \tag{2}$$

where $\phi^{(s_1,s_2)} \equiv \frac{\partial^{s_1+s_2}\phi}{\partial x_1^{s_1}\partial x_2^{s_2}}$ is assumed to exist. For reasons given below (see (8)), the functions in $\mathcal{U}_{\underline{s}-cx}^{[\underline{a},\underline{b}]}$ are called regular $\underline{s}$-convex. Putting

$$\mathcal{K}(\underline{s}) = \Big\{\underline{k} \in \mathbb{N}^2 \mid \underline{0} \le \underline{k} \le \underline{s},\ \ k_1 + k_2 \ge 1\Big\},$$

$\mathcal{U}_{\underline{s}-icx}^{[\underline{a},\underline{b}]}$ denotes the subfamily of the regular $\underline{s}$-increasing convex functions defined as

$$\begin{aligned} \mathcal{U}_{\underline{s}-icx}^{[\underline{a},\underline{b}]} &= \bigcap_{\underline{k}\in\mathcal{K}(\underline{s})} \mathcal{U}_{\underline{k}-cx}^{[\underline{a},\underline{b}]} \\ &= \Big\{\phi : [\underline{a},\underline{b}] \to \mathbb{R} \mid \phi^{(k_1,k_2)} \ge 0 \text{ on } [\underline{a},\underline{b}],\ \text{for all } \underline{k} \in \mathcal{K}(\underline{s})\Big\}. \end{aligned} \tag{3}$$

Then, $\underline{X}$ is said to be smaller than $\underline{Y}$ in the $\underline{s}$-increasing convex (resp. $\underline{s}$ convex) ordering, which is denoted by $\underline{X}\preceq_{\underline{s}-icx}^{[\underline{a},\underline{b}]}\underline{Y}$ (resp. $\underline{X}\preceq_{\underline{s}-cx}^{[\underline{a},\underline{b}]}\underline{Y}$), when (1) holds with $\mathcal{F} = \mathcal{U}_{\underline{s}-icx}^{[\underline{a},\underline{b}]}$ (resp. $\mathcal{U}_{\underline{s}-cx}^{[\underline{a},\underline{b}]}$).

The notion of convex functions in the sense of Popoviciu [12] is more general than that defined in (2). Let us first recall the definition of the divided difference operator. In the univariate case, given a function $\phi : [a,b] \to \mathbb{R}$ and points $x_0 < x_1 < \ldots < x_s \in [a,b]$, with $s \in \mathbb{N}$, this operator is defined recursively by $[x_i]\phi = \phi(x_i)$, $i = 0,1,\ldots,s$, and

$$[x_0,\ldots,x_s]\phi = \frac{[x_1,\ldots,x_s]\phi - [x_0,\ldots,x_{s-1}]\phi}{x_s - x_0}. \tag{4}$$

In the bivariate case, given a function $\phi : [\underline{a},\underline{b}] \to \mathbb{R}$ and points $x_0 < x_1 < \ldots < x_{s_1} \in [a_1,b_1]$ and $y_0 < y_1 < \ldots < y_{s_2} \in [a_2,b_2]$, $\underline{s} \in \mathbb{N}^2$, the partial divided difference operator is defined by

$$\begin{aligned} \begin{bmatrix} x_0,\ldots,x_{s_1} \\ y_0,\ldots,y_{s_2} \end{bmatrix}\phi &= [x_0,\ldots,x_{s_1}]([y_0,\ldots,y_{s_2}]\phi) \\ &= [y_0,\ldots,y_{s_2}]([x_0,\ldots,x_{s_1}]\phi). \end{aligned} \tag{5}$$

Then, $\phi : [\underline{a}, \underline{b}] \to \mathbb{R}$ is $\underline{s}$-convex, $\underline{s} \in \mathbb{N}^2$, if the partial divided differences (5) are non-negative for all points $x_0 < x_1 < \ldots < x_{s_1} \in [a_1, b_1]$ and $y_0 < y_1 < \ldots < y_{s_2} \in [a_2, b_2]$. We denote by $\overline{\mathcal{U}}_{\underline{s}-cx}^{[\underline{a},\underline{b}]}$, $\underline{s} \in \mathbb{N}^2$, the family of the continuous $\underline{s}$-convex functions, i.e.

$$\overline{\mathcal{U}}_{\underline{s}-cx}^{[\underline{a},\underline{b}]} = \Big\{ \phi : [\underline{a}, \underline{b}] \to \mathbb{R} \mid \phi \text{ is continuous and } \begin{bmatrix} x_0, \ldots, x_{s_1} \\ y_0, \ldots, y_{s_2} \end{bmatrix} \phi \geq 0,$$

$$\text{for all points } x_0 < \ldots < x_{s_1} \in [a_1, b_1] \text{ and } y_0 < \ldots < y_{s_2} \in [a_2, b_2] \Big\}. \tag{6}$$

As above, the family $\overline{\mathcal{U}}_{\underline{s}-icx}^{[\underline{a},\underline{b}]}$ of the continuous $\underline{s}$-increasing convex functions is defined by

$$\overline{\mathcal{U}}_{\underline{s}-icx}^{[\underline{a},\underline{b}]} = \bigcap_{\underline{k} \in \mathcal{K}(\underline{s})} \overline{\mathcal{U}}_{\underline{k}-cx}^{[\underline{a},\underline{b}]}. \tag{7}$$

It is well-known that if $\phi^{(s_1,s_2)}$ exists, then

$$\phi \in \overline{\mathcal{U}}_{\underline{s}-cx}^{[\underline{a},\underline{b}]} \Leftrightarrow \phi^{(s_1,s_2)} \geq 0, \quad \text{on } [\underline{a}, \underline{b}], \tag{8}$$

but a function ϕ in $\overline{\mathcal{U}}_{\underline{s}-cx}^{[\underline{a},\underline{b}]}$ has not necessarily a partial derivative $\phi^{(s_1,s_2)}$ (although the $\underline{s}$-convexity implies certain regularity properties); therefore,

$$\mathcal{U}_{\underline{s}-cx}^{[\underline{a},\underline{b}]} \subset \overline{\mathcal{U}}_{\underline{s}-cx}^{[\underline{a},\underline{b}]} \quad \text{and} \quad \mathcal{U}_{\underline{s}-icx}^{[\underline{a},\underline{b}]} \subset \overline{\mathcal{U}}_{\underline{s}-icx}^{[\underline{a},\underline{b}]}. \tag{9}$$

Moreover, any function ϕ in $\overline{\mathcal{U}}_{\underline{s}-icx}^{[\underline{a},\underline{b}]}$, for some $\underline{s} \geq \underline{2}$, is continuous on $[\underline{a}, \underline{b}]$; this is not true, however, when s_1 or $s_2 = 1$.

We now introduce the following family of bivariate functions, denoted by $\underline{\mathcal{U}}_{\underline{s}-icx}^{[\underline{a},\underline{b}]}$, for $\underline{s} \geq \underline{1}$:

$$\begin{aligned} \underline{\mathcal{U}}_{\underline{s}-icx}^{[\underline{a},\underline{b}]} = \Big\{ & (x_1 - a_1)^{i_1} (x_2 - a_2)^{i_2}, \ \underline{0} \leq \underline{i} \leq \underline{s} - \underline{1}; \\ & (x_1 - a_1)^{i_1} (x_2 - t_2)_+^{s_2 - 1}, \ 0 \leq i_1 \leq s_1 - 1, \ t_2 \in [a_2, b_2]; \\ & (x_1 - t_1)_+^{s_1 - 1} (x_2 - a_2)^{i_2}, \ 0 \leq i_2 \leq s_2 - 1, \ t_1 \in [a_1, b_1]; \\ & (x_1 - t_1)_+^{s_1 - 1} (x_2 - t_2)_+^{s_2 - 1}, \ \underline{t} \in [\underline{a}, \underline{b}] \Big\}. \end{aligned} \tag{10}$$

Note that the functions in (10) have a very simple product form. It is easily seen that these functions are continuous $\underline{s}$-increasing convex, i.e.

$$\underline{\mathcal{U}}_{\underline{s}-icx}^{[\underline{a},\underline{b}]} \subset \overline{\mathcal{U}}_{\underline{s}-cx}^{[\underline{a},\underline{b}]}. \tag{11}$$

Coming back to the notion of integral stochastic ordering $\preceq_{\mathcal{F}}^{[\underline{a},\underline{b}]}$, it is interesting, from a theoretical point of view as well as for certain applications, to substitute for the generating cone $\mathcal{F}$, either a dense subfamily of functions contained in $\mathcal{F}$, or a larger family corresponding to the closure of $\mathcal{F}$ in some topology. The smallest and largest such classes, $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ say, are called the minimal and maximal generators (Müller [10]). Hereafter, we aim to establish that for the $\preceq^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$ ordering, $\underline{s} \geq \underline{1}$, the extremal generators are $\underline{\mathcal{F}} = \underline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$ and $\overline{\mathcal{F}} = \overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$. The proof for $\underline{\mathcal{F}}$ will be immediate from a known approximation property. For $\overline{\mathcal{F}}$, the result will follow directly by showing that any function ϕ in $\overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$ can be uniformly approximated by appropriate spline functions.

3. SPLINE APPROXIMATION FOR FUNCTIONS IN $\overline{\mathcal{U}}^{[\mathbf{\underline{a}},\mathbf{\underline{b}}]}_{\mathbf{\underline{s}-icx}}$

For simplicity, we consider the particular interval $[\underline{0},\underline{1}]$; the general case $[\underline{a},\underline{b}]$ follows by straightforward substitutions.

We are going to establish that the family of functions $\underline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$ is dense in $\overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$, $\underline{s} \geq \underline{1}$. For that, we will mainly apply to the bivariate case an argument of Bojanic and Roulier [2], and which is based on an intermediate uniform approximation of any function ϕ in $\overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$, $\underline{s} \geq \underline{2}$, by Bernstein polynomials.

In the sequel, any partial derivative which will be used is assumed to exist.

Lemma 3.1. *Let $\phi : [\underline{0},\underline{1}] \to \mathbb{R}$ be a function such that*

$$\phi^{(i_1,s_2)} \geq 0, \quad \textit{for } i_1 = 0,\dots,s_1 \quad \textit{and} \quad \phi^{(s_1,i_2)} \geq 0, \quad \textit{for } i_2 = 0,\dots,s_2,$$

for some $\underline{s} \geq \underline{2}$. Put

$$\Omega_{\underline{s}-\underline{1}}(\phi;\underline{x}) = \sum_{i_1=0}^{s_1-1}\sum_{i_2=0}^{s_2-1} \phi^{(i_1,i_2)}(0,0)\frac{x_1^{i_1}x_2^{i_2}}{i_1!\, i_2!}, \tag{12}$$

as the Taylor polynomial of ϕ of degree (s_1-1, s_2-1). Then, for every $n \geq 2$, there exist in $[0,1]$ the points $\alpha_1^{(i_1)} < \ldots < \alpha_{n-1}^{(i_1)}$, $i_1 = 0,\dots,s_1-2$, the points $\beta_1^{(i_2)} < \ldots < \beta_{n-1}^{(i_2)}$, $i_2 = 0,\dots,s_2-2$, the points $\epsilon_1 < \ldots < \epsilon_{n-1}$ and the points $\vartheta_1 < \ldots < \vartheta_{n-1}$ such that

$$\begin{aligned}
\phi(\underline{x}) \;=\; & \Omega_{\underline{s}-\underline{1}}(\phi;\underline{x}) + \sum_{i_1=0}^{s_1-2}\tfrac{\phi^{(i_1,s_2-1)}(0,1)-\phi^{(i_1,s_2-1)}(0,0)}{i_1!\,(s_2-1)!\,n}\sum_{k=1}^{n-1}x_1^{i_1}(x_2-\alpha_k^{(i_1)})_+^{s_2-1} \\
& + \sum_{i_2=0}^{s_2-2}\tfrac{\phi^{(s_1-1,i_2)}(1,0)-\phi^{(s_1-1,i_2)}(0,0)}{i_2!\,(s_1-1)!\,n}\sum_{k=1}^{n-1}x_2^{i_2}(x_1-\beta_k^{(i_2)})_+^{s_1-1} \qquad (13)\\
& + \frac{\phi^{(s_1-1,s_2-1)}(1,1)-\phi^{(s_1-1,s_2-1)}(0,0)}{(s_1-1)!\,(s_2-1)!\,n^2}\sum_{\ell=1}^{n-1}\sum_{k=1}^{n-1}(x_1-\epsilon_\ell)_+^{s_1-1}(x_2-\vartheta_k)_+^{s_2-1} \\
& + R_1^{(n)}(\phi;\underline{x}) + R_2^{(n)}(\phi;\underline{x}) + R^{(n)}(\phi;\underline{x}),
\end{aligned}$$

where

$$(14)\quad \left|R_1^{(n)}(\phi;\underline{x})\right| \le \tfrac{e}{(s_2-1)!\; n} \sup_{i_1\le s_1-2} \int_{t_2=0}^{1} \phi^{(i_1,s_2)}(0,t_2)\mathrm{d}t_2,$$

$$(15)\quad \left|R_2^{(n)}(\phi;\underline{x})\right| \le \tfrac{e}{(s_1-1)!\; n} \sup_{i_2\le s_2-2} \int_{t_1=0}^{1} \phi^{(s_1,i_2)}(t_1,0)\mathrm{d}t_1,$$

$$(16)\quad \left|R^{(n)}(\phi;\underline{x})\right| \le$$

$$\le \tfrac{2}{(s_1-1)!\;(s_2-1)!\; n}\Big\{ \int_{t_2=0}^{1} \phi^{(s_1-1,s_2)}(0,t_2)\mathrm{d}t_2 + \int_{t_1=0}^{1} \phi^{(s_1,s_2-1)}(t_1,0)\mathrm{d}t_1 \Big\}.$$

Proof. By Taylor's expansion of $\phi(x_1,x_2)$ viewed as a function of x_1 around 0 (for fixed x_2), we have

$$(17)\qquad \phi(\underline{x}) = \sum_{i_1=0}^{s_1-2} \phi^{(i_1,0)}(0,x_2)\tfrac{x_1^{i_1}}{i_1!} + \int_{t_1=0}^{x_1} \tfrac{(x_1-t_1)^{s_1-2}}{(s_1-2)!}\phi^{(s_1-1,0)}(t_1,x_2)\mathrm{d}t_1.$$

Inserting in (17) Taylor's expansions of $\phi^{(i_1,0)}(0,x_2)$ and $\phi^{(s_1-1,0)}(t_1,x_2)$ as functions of x_2, i.e.

$$\phi^{(i_1,0)}(0,x_2) = \sum_{i_2=0}^{s_2-2} \phi^{(i_1,i_2)}(0,0)\tfrac{x_2^{i_2}}{i_2!} + \int_{t_2=0}^{x_2} \tfrac{(x_2-t_2)^{s_2-2}}{(s_2-2)!}\phi^{(i_1,s_2-1)}(0,t_2)\mathrm{d}t_2,$$

and

$$\phi^{(s_1-1,0)}(t_1,x_2) = \sum_{i_2=0}^{s_2-2} \phi^{(s_1-1,i_2)}(t_1,0)\tfrac{x_2^{i_2}}{i_2!} + \int_{t_2=0}^{x_2} \tfrac{(x_2-t_2)^{s_2-2}}{(s_2-2)!}\phi^{(s_1-1,s_2-1)}(t_1,t_2)\mathrm{d}t_2,$$

we obtain

$$\begin{aligned}\phi(\underline{x}) = {} & \sum_{i_1=0}^{s_1-2}\sum_{i_2=0}^{s_2-2} \phi^{(i_1,i_2)}(0,0)\tfrac{x_1^{i_1}x_2^{i_2}}{i_1!\; i_2!} + \sum_{i_1=0}^{s_1-2} \int_{t_2=0}^{1} \tfrac{x_1^{i_1}(x_2-t_2)_+^{s_2-2}}{i_1!(s_2-2)!}\phi^{(i_1,s_2-1)}(0,t_2)\mathrm{d}t_2 \\ & + \sum_{i_2=0}^{s_2-2} \int_{t_1=0}^{1} \tfrac{x_2^{i_2}(x_1-t_1)_+^{s_1-2}}{i_2!(s_1-2)!}\phi^{(s_1-1,i_2)}(t_1,0)\mathrm{d}t_1 \\ & + \int_{t_1=0}^{1}\int_{t_2=0}^{1} \tfrac{(x_1-t_1)_+^{s_1-2}(x_2-t_2)_+^{s_2-2}}{(s_1-2)!\;(s_2-2)!}\phi^{(s_1-1,s_2-1)}(t_1,t_2)\mathrm{d}t_2\mathrm{d}t_1\end{aligned}$$

and thus, using (12),

$$\begin{aligned}
\phi(\underline{x}) =&\Omega_{\underline{s}-\underline{1}}(\phi;\underline{x})\\
&+\sum_{i_1=0}^{s_1-2}\int_{t_2=0}^{1}\tfrac{x_1^{i_1}(x_2-t_2)_+^{s_2-2}}{i_1!(s_2-2)!}\Big\{\phi^{(i_1,s_2-1)}(0,t_2)-\phi^{(i_1,s_2-1)}(0,0)\Big\}\,\mathrm{d}t_2\\
&+\sum_{i_2=0}^{s_2-2}\int_{t_1=0}^{1}\tfrac{x_2^{i_2}(x_1-t_1)_+^{s_1-2}}{i_2!(s_1-2)!}\Big\{\phi^{(s_1-1,i_2)}(t_1,0)-\phi^{(s_1-1,i_2)}(0,0)\Big\}\,\mathrm{d}t_1\\
&+\int_{t_1=0}^{1}\int_{t_2=0}^{1}\Big[\tfrac{(x_1-t_1)_+^{s_1-2}(x_2-t_2)_+^{s_2-2}}{(s_1-2)!\,(s_2-2)!}\{\phi^{(s_1-1,s_2-1)}(t_1,t_2)-\\
&\qquad\qquad -\phi^{(s_1-1,s_2-1)}(0,0)\}\Big]\mathrm{d}t_2\mathrm{d}t_1.
\end{aligned}\tag{18}$$

Let us look at the three kinds of functions inside $\{\ldots\}$ in (18). Since by hypothesis, $\phi^{(i_1,s_2)}\geq 0, i_1=0,\ldots,s_1$, each function $t_2\mapsto\phi^{(i_1,s_2-1)}(0,t_2)-\phi^{(i_1,s_2-1)}(0,0), i_1=0,\ldots,s_1-2$, is non-negative, non-decreasing and continuous on $[0,1]$. Thus, it can be approximated by a step function of the form

$$\psi_{i_1,1}^{(n)}(\phi;t_2)\equiv\frac{\phi^{(i_1,s_2-1)}(0,1)-\phi^{(i_1,s_2-1)}(0,0)}{n}\sum_{k=1}^{n-1}(t_2-\alpha_k^{(i_1)})_+^0,\tag{19}$$

for some points $\alpha_1^{(i_1)}<\ldots<\alpha_{n-1}^{(i_1)}$ in $[0,1]$. Moreover, the constants $\alpha_k^{(i_1)}$ can be chosen such that

$$\begin{aligned}
|\rho_{i_1,1}^{(n)}(\phi;t_2)| &\equiv \Big|\phi^{(i_1,s_2-1)}(0,t_2)-\phi^{(i_1,s_2-1)}(0,0)-\psi_{i_1,1}^{(n)}(\phi;t_2)\Big|\\
&\leq \frac{\phi^{(i_1,s_2-1)}(0,1)-\phi^{(i_1,s_2-1)}(0,0)}{n}\\
&= \tfrac{1}{n}\int_{t_2=0}^{1}\phi^{(i_1,s_2)}(0,t_2)\mathrm{d}t_2.
\end{aligned}\tag{20}$$

Similarly, one can find some points $\beta_1^{(i_2)}<\ldots<\beta_{n-1}^{(i_2)}$, $i_2=0,\ldots,s_2-1$ in $[0,1]$ such that

$$\psi_{i_2,2}^{(n)}(\phi;t_1)\equiv\frac{\phi^{(s_1-1,i_2)}(1,0)-\phi^{(s_1-1,i_2)}(0,0)}{n}\sum_{k=1}^{n-1}(t_1-\beta_k^{(i_2)})_+^0,\tag{21}$$

satisfies

$$\begin{aligned}
|\rho_{i_2,2}^{(n)}(\phi;t_1)| &\equiv \Big|\phi^{(s_1-1,i_2)}(t_1,0)-\phi^{(s_1-1,i_2)}(0,0)-\psi_{i_2,2}^{(n)}(\phi;t_1)\Big|\\
&\leq \frac{\phi^{(s_1-1,i_2)}(1,0)-\phi^{(s_1-1,i_2)}(0,0)}{n}=\\
&= \tfrac{1}{n}\int_{t_1=0}^{1}\phi^{(s_1,i_2)}(t_1,0)\mathrm{d}t_1.
\end{aligned}\tag{22}$$

Finally, given any points $\epsilon_1 < \ldots < \epsilon_{n-1}$ and $\vartheta_1 < \ldots < \vartheta_{n-1}$ in $[0,1]$, let us consider the function

(23)
$$\psi^{(n)}(\phi;\underline{t}) \equiv \frac{\phi^{(s_1-1,s_2-1)}(1,1)-\phi^{(s_1-1,s_2-1)}(0,0)}{n^2}\sum_{\ell=1}^{n-1}\sum_{k=1}^{n-1}(t_1-\epsilon_\ell)_+^0(t_2-\vartheta_k)_+^0,$$

and put

$$\left|\rho^{(n)}(\phi;\underline{t})\right| \equiv \left|\phi^{(s_1-1,s_2-1)}(t_1,t_2)-\phi^{(s_1-1,s_2-1)}(0,0)-\psi^{(n)}(\phi;\underline{t})\right|. \tag{24}$$

For $\underline{t} \in [\epsilon_\ell,\epsilon_{\ell+1}[\times[\vartheta_k,\vartheta_{k+1}[$, we get

$$\psi^{(n)}(\phi;\underline{t}) = \frac{\phi^{(s_1-1,s_2-1)}(1,1)-\phi^{(s_1-1,s_2-1)}(0,0)}{n^2}\ell k. \tag{25}$$

But $\phi^{(s_1-1,s_2-1)}(\underline{t})$ being continuous and non-decreasing on $[\underline{0},\underline{1}]$ by hypothesis, we have that for such $\underline{t}$,

$$\phi^{(s_1-1,s_2-1)}(\epsilon_\ell,\vartheta_k) \le \phi^{(s_1-1,s_2-1)}(t_1,t_2) \le \phi^{(s_1-1,s_2-1)}(\epsilon_{\ell+1},\vartheta_{k+1}); \tag{26}$$

furthermore, we can choose the ϵ_ℓ's and ϑ_k's in such a way that

$$\begin{aligned} &\phi^{(s_1-1,s_2-1)}(\epsilon_\ell,\vartheta_k)-\phi^{(s_1-1,s_2-1)}(0,0) = \\ &= \frac{\phi^{(s_1-1,s_2-1)}(1,1)-\phi^{(s_1-1,s_2-1)}(0,0)}{n^2}\ell k. \end{aligned} \tag{27}$$

Thus, combining (25), (26), (27) with (24), we obtain that, for $\underline{t} \in [\epsilon_\ell,\epsilon_{\ell+1}[\times[\vartheta_k,\vartheta_{k+1}[$,

$$\left|\rho^{(n)}(\phi;\underline{t})\right| \le \frac{\phi^{(s_1-1,s_2-1)}(1,1)-\phi^{(s_1-1,s_2-1)}(0,0)}{n^2}\{(\ell+1)(k+1)-\ell k\},$$

which yields, for all $\underline{t}$ in $[\underline{0},\underline{1}]$,

$$\begin{aligned} \left|\rho^{(n)}(\phi;\underline{t})\right| &\le \tfrac{2}{n}\left\{\phi^{(s_1-1,s_2-1)}(1,1)-\phi^{(s_1-1,s_2-1)}(0,0)\right\} \\ &= \tfrac{2}{n}\left\{\int_{t_2=0}^{1}\phi^{(s_1-1,s_2)}(1,t_2)\mathrm{d}t_2+\int_{t_1=0}^{1}\phi^{(s_1,s_2-1)}(t_1,0)\mathrm{d}t_1\right\}. \end{aligned} \tag{28}$$

Now, using the definitions of $\rho_{i_1,1}^{(n)}(\phi;t_2),\rho_{i_2,2}^{(n)}(\phi;t_1)$ and $\rho^{(n)}(\phi;\underline{t})$, we see that (18) can be rewritten exactly as (13) in which

$$R_1^{(n)}(\phi;\underline{x}) = \sum_{i_1=0}^{s_1-2}\int_{t_2=0}^{1}\tfrac{x_1^{i_1}(x_2-t_2)_+^{s_2-2}}{i_1!(s_2-2)!}\rho_{i_1,1}^{(n)}(\phi;t_2)\mathrm{d}t_2,$$

$$\begin{aligned} R_2^{(n)}(\phi;\underline{x}) &= \sum_{i_2=0}^{s_2-2}\int_{t_1=0}^{1}\tfrac{x_2^{i_2}(x_1-t_1)_+^{s_1-2}}{i_2!(s_1-2)!}\rho_{i_2,2}^{(n)}(\phi;t_1)\mathrm{d}t_1, \\ R^{(n)}(\phi;\underline{x}) &= \int_{t_1=0}^{1}\int_{t_2=0}^{1}\tfrac{(x_1-t_1)_+^{s_1-2}(x_2-t_2)_+^{s_2-2}}{(s_1-2)!(s_2-2)!}\rho^{(n)}(\phi;\underline{t})\mathrm{d}t_1\mathrm{d}t_2. \end{aligned}$$

Therefore, it remains to verify that the majorizations (14), (15) and (16) are valid. From (20), we find that

$$\begin{aligned}
&|R_1^{(n)}(\phi;\underline{x})| \leq \\
&\leq \sum_{i_1=0}^{s_1-2} \int_{t_2=0}^{1} \tfrac{x_1^{i_1}(x_2-t_2)_+^{s_2-2}}{i_1!(s_2-2)!} |\rho_{1,i_1}^{(n)}(\phi;t_2)| \mathrm{d}t_2 \\
&\leq \tfrac{1}{n} \sup_{i_1\leq s_1-2} \left\{ \int_{t_2=0}^{1} \phi^{(i_1,s_2)}(0,t_2)\mathrm{d}t_2 \right\} \sum_{i_1=0}^{s_1-2} \tfrac{x_1^{i_1}}{i_1!(s_2-2)!} \int_{t_2=0}^{1} (x_2-t_2)_+^{s_2-2}\mathrm{d}t_2 \\
&= \tfrac{1}{n} \sup_{i_1\leq s_1-2} \left\{ \int_{t_2=0}^{1} \phi^{(i_1,s_2)}(0,t_2)\mathrm{d}t_2 \right\} \sum_{i_1=0}^{s_1-2} \tfrac{x_1^{i_1}x_2^{s_2-1}}{i_1!(s_2-1)!},
\end{aligned}$$

hence (14). In the same way, (15) follows from (22) and (16) follows from (28). □

Given any continuous function $\phi : [\underline{0},\underline{1}] \to \mathbb{R}$, let $B_{\underline{m}}(\phi,.)$ denote the Bernstein polynomial of ϕ of degree $\underline{m} \in \mathbb{N}^2$, that is,

$$\begin{aligned}
(29)\qquad &B_{\underline{m}}(\phi;\underline{x}) = \\
&= \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} \tbinom{m_1}{k_1}\tbinom{m_2}{k_2}\phi(\tfrac{k_1}{m_1},\tfrac{k_2}{m_2}) x_1^{k_1}(1-x_1)^{m_1-k_1} x_2^{k_2}(1-x_2)^{m_2-k_2}.
\end{aligned}$$

In the next lemma, we show that the Bernstein polynomial of any $\underline{s}$-increasing convex function is also a (regular) $\underline{s}$-increasing convex function. This property is the bivariate extension of a classical result by Popoviciu [13].

Hereafter, we will have recourse to the following standard operator (see, e.g., Agarwal [1]); given $\underline{\ell} \in \mathbb{N}^2$ and h_1, $h_2 > 0$, let

$$(30)\qquad \Delta_{h_1,h_2}^{(\ell_1,\ell_2)}\phi(a_1,a_2) = \ell_1!\ell_2! h_1^{\ell_1} h_2^{\ell_2} \left[\begin{array}{cccc} a_1, & a_1+h_1, & \ldots, & a_1+\ell_1 h_1 \\ a_2, & a_2+h_2, & \ldots, & a_2+\ell_2 h_2 \end{array} \right] \phi.$$

Lemma 3.2. *If a function ϕ belongs to $\overline{\mathcal{U}}_{\underline{s}-icx}^{[\underline{0},\underline{1}]}$, then the polynomial $B_{\underline{m}}(\phi;.)$, $\underline{m} \geq \underline{s}$, belongs to $\mathcal{U}_{\underline{s}-icx}^{[\underline{0},\underline{1}]}$.*

Proof. We are going to establish that for all $\underline{0} \leq \underline{\ell} \leq \underline{m}$, the derivative $\{B_{\underline{m}}(\phi;.)\}^{(\ell_1,\ell_2)}$ can be expressed as

$$\begin{aligned}
\{B_{\underline{m}}(\phi;\underline{x})\}^{(\ell_1,\ell_2)} \;=\; &\tfrac{m_1!m_2!}{(m_1-\ell_1)!(m_2-\ell_2)!} \sum_{k_1=0}^{m_1-\ell_1} \sum_{k_2=0}^{m_2-\ell_2} \Delta_{\frac{1}{m_1},\frac{1}{m_2}}^{(\ell_1,\ell_2)} \phi(\tfrac{k_1}{m_1},\tfrac{k_2}{m_2}) \tbinom{m_1-\ell_1}{k_1} \cdot \\
(31)\qquad &\cdot \tbinom{m_2-\ell_2}{k_2} x_1^{k_1}(1-x_1)^{m_1-\ell_1-k_1} x_2^{k_2}(1-x_2)^{m_2-\ell_2-k_2}.
\end{aligned}$$

By (3), this leads directly to the announced property. Obviously, (31) is true for $\ell_1 = \ell_2 = 0$. Thus, let us proceed by induction. We then get

$$\begin{aligned}
&\{B_{\underline{m}}(\phi;\underline{x})\}^{(\ell_1+1,\ell_2)} = \\
&= \tfrac{m_1!m_2!}{(m_1-\ell_1)!(m_2-\ell_2)!} \cdot \\
&\cdot \Bigg\{ \sum_{k_1=1}^{m_1-\ell_1} \sum_{k_2=0}^{m_2-\ell_2} \Big\{ \Delta^{(\ell_1,\ell_2)}_{\frac{1}{m_1},\frac{1}{m_2}} \phi\big(\tfrac{k_1}{m_1},\tfrac{k_2}{m_2}\big) \tbinom{m_1-\ell_1}{k_1}\tbinom{m_2-\ell_2}{k_2} k_1 x_1^{k_1-1} \cdot \\
&\qquad \cdot (1-x_1)^{m_1-\ell_1-k_1} x_2^{k_2} (1-x_2)^{m_2-\ell_2-k_2} \Big\} - \\
&- \sum_{k_1=0}^{m_1-\ell_1-1} \sum_{k_2=0}^{m_2-\ell_2} \Big\{ \Delta^{(\ell_1,\ell_2)}_{\frac{1}{m_1},\frac{1}{m_2}} \phi\big(\tfrac{k_1}{m_1},\tfrac{k_2}{m_2}\big) \tbinom{m_1-\ell_1}{k_1}\tbinom{m_2-\ell_2}{k_2} x_1^{k_1} (m_1-\ell_1-k_1) \cdot \\
&\qquad \cdot (1-x_1)^{m_1-\ell_1-k_1-1} x_2^{k_2} (1-x_2)^{m_2-\ell_2-k_2} \Big\} \Bigg\},
\end{aligned}$$

yielding

$$\begin{aligned}
&\{B_{\underline{m}}(\phi;\underline{x})\}^{(\ell_1+1,\ell_2)} = \\
&\qquad = \tfrac{m_1!\; m_2!}{(m_1-\ell_1)!\;(m_2-\ell_2)!} \sum_{k_1=0}^{m_1-\ell_1-1} \sum_{k_2=0}^{m_2-\ell_2} \tbinom{m_1-\ell_1}{k_1}\tbinom{m_2-\ell_2}{k_2} \cdot \\
&\qquad \cdot (m_1-\ell_1-k_1) x_1^{k_1} (1-x_1)^{m_1-\ell_1-k_1-1} x_2^{k_2} (1-x_2)^{m_2-\ell_2-k_2} \cdot \\
&\qquad \cdot \Big\{ \Delta^{(\ell_1,\ell_2)}_{\frac{1}{m_1},\frac{1}{m_2}} \phi\big(\tfrac{k_1+1}{m_1},\tfrac{k_2}{m_2}\big) - \Delta^{(\ell_1,\ell_2)}_{\frac{1}{m_1},\frac{1}{m_2}} \phi\big(\tfrac{k_1}{m_1},\tfrac{k_2}{m_2}\big) \Big\}. \qquad (32)
\end{aligned}$$

But we notice that by (30) and (5),

$$\begin{aligned}
&\Delta^{(\ell_1,\ell_2)}_{\frac{1}{m_1},\frac{1}{m_2}} \phi\big(\tfrac{k_1+1}{m_1},\tfrac{k_2}{m_2}\big) - \Delta^{(\ell_1,\ell_2)}_{\frac{1}{m_1},\frac{1}{m_2}} \phi\big(\tfrac{k_1}{m_1},\tfrac{k_2}{m_2}\big) = \\
&\quad = \ell_1!\;\ell_2!\big(\tfrac{1}{m_1}\big)^{\ell_1}\big(\tfrac{1}{m_2}\big)^{\ell_2} \left(\begin{bmatrix} \frac{k_1+1}{m_1}, & \frac{k_1+2}{m_1}, & \ldots, & \frac{k_1+\ell_1+1}{m_1} \\ \frac{k_2}{m_2}, & \frac{k_2+1}{m_2}, & \ldots, & \frac{k_2+\ell_2}{m_2} \end{bmatrix}\phi \right. \\
&\qquad\qquad \left. - \begin{bmatrix} \frac{k_1}{m_1}, & \frac{k_1+1}{m_1}, & \ldots, & \frac{k_1+\ell_1}{m_1} \\ \frac{k_2}{m_2}, & \frac{k_2+1}{m_2}, & \ldots, & \frac{k_2+\ell_2}{m_2} \end{bmatrix}\phi \right) = \\
&= \ell_1!\;\ell_2!\big(\tfrac{1}{m_1}\big)^{\ell_1}\big(\tfrac{1}{m_2}\big)^{\ell_2} \tfrac{\ell_1+1}{m_1} \begin{bmatrix} \frac{k_1}{m_1}, & \frac{k_1+1}{m_1}, & \ldots, & \frac{k_1+\ell_1+1}{m_1} \\ \frac{k_2}{m_2}, & \frac{k_2+1}{m_2}, & \ldots, & \frac{k_2+\ell_2}{m_2} \end{bmatrix}\phi \\
&= \Delta^{(\ell_1+1,\ell_2)}_{\frac{1}{m_1},\frac{1}{m_2}} \phi\big(\tfrac{k_1}{m_1},\tfrac{k_2}{m_2}\big),
\end{aligned}$$

so that (32) reduces to the form (31). □

It is well-known that any continuous function $\phi : [\underline{0},\underline{1}] \to \mathbb{R}$ can be approximated uniformly by a sequence of Bernstein polynomials $B_{\underline{m}}(\phi,.)$ as

$\min(m_1, m_2) \to \infty$ (see, e.g., Lorentz [7]). Now, given any function $\phi \in \overline{\mathcal{U}}^{[\underline{0},\underline{1}]}_{\underline{s}-icx}$, $\underline{s} \geq \underline{1}$, we are going to build a sequence of spline functions $\psi^{(n)}_{\underline{m}}(\phi, .)$ that are non-negative linear combinations of functions in $\underline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$, and that converge uniformly to the polynomial $B_{\underline{m}}(\phi, .)$ as $n \to \infty$. Combining both approximations will then provide a uniform approximation of $\phi \in \overline{\mathcal{U}}^{[\underline{0},\underline{1}]}_{\underline{s}-icx}$ by the functions $\psi^{(n)}_{\underline{m}}(\phi, .)$. This will show also that $\underline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$ is dense in $\overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$, $\underline{s} \geq \underline{1}$.

For clarity, the precise statement is given with respect to a general interval $[\underline{a}, \underline{b}]$ (instead of $[\underline{0}, \underline{1}]$).

Proposition 3.3. *Every function $\phi \in \overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$, $\underline{s} \geq \underline{2}$, can be approximated uniformly on $[\underline{a}, \underline{b}]$, as $n \to \infty$, by spline functions $\psi^{(n)}_{\underline{m}}(\phi, .)$, $\underline{m} \geq \underline{s}$ and $n \geq 2$, which are of the form*

$$
\begin{aligned}
\psi^{(n)}_{\underline{m}}(\phi, \underline{x}) \;=\; & \sum_{k_1=0}^{s_1-1}\sum_{k_2=0}^{s_2-1} \tbinom{s_1}{k_1}\tbinom{s_2}{k_2} \Delta^{(k_1,k_2)}_{\frac{b_1-a_1}{m_1},\frac{b_2-a_2}{m_2}} \phi(a_1, a_2) \tfrac{(x_1-a_1)^{k_1}(x_2-a_2)^{k_2}}{(b_1-a_1)^{k_1}(b_2-a_2)^{k_2}} \\
& + \sum_{i_1=0}^{s_1-2} \frac{C_{i_1,1}(\phi, \underline{m})}{n} \sum_{k=1}^{n-1} \frac{(x_2 - \alpha^{(i_1)}_k)^{s_2-1}_+ (x_1 - a_1)^{i_1}}{(b_2 - a_2)^{s_2-1}(b_1 - a_1)^{i_1}} \\
& + \sum_{i_2=0}^{s_2-2} \frac{C_{i_2,2}(\phi, \underline{m})}{n} \sum_{k=1}^{n-1} \frac{(x_1 - \beta^{(i_2)}_k)^{s_1-1}_+ (x_2 - a_2)^{i_2}}{(b_1 - a_1)^{s_1-1}(b_2 - a_2)^{i_2}} \\
& + \frac{C(\phi; \underline{m})}{n^2} \sum_{\ell=1}^{n-1}\sum_{k=1}^{n-1} \frac{(x_1 - \epsilon_\ell)^{s_1-1}_+ (x_2 - \vartheta_k)^{s_2-1}_+}{(b_1 - a_1)^{s_1-1}(b_2 - a_2)^{s_2-1}},
\end{aligned}
\tag{33}
$$

for some points $\alpha^{(i_1)}_1 < \ldots < \alpha^{(i_1)}_{n-1}$, $i_1 = 0, \ldots, s_1 - 2$, and $\epsilon_1 < \ldots < \epsilon_{n-1}$ in $[a_1, b_1]$, and some points $\beta^{(i_2)}_1 < \ldots < \beta^{(i_2)}_{n-1}$, $i_2 = 0, \ldots, s_2 - 2$, and $\vartheta_1 < \ldots < \vartheta_{n-1}$ in $[a_2, b_2]$, and where the constants $C_{i_1,1}(.)$, $C_{i_2,2}(.)$ and $C(.)$ are non-negative and given by

$$
C_{i_1,1}(\phi, \underline{m}) = \tbinom{m_1}{i_1}\tbinom{m_2}{s_2-1} \sum_{k_2=0}^{m_2-s_2} \Delta^{(i_1,s_2)}_{\frac{b_1-a_1}{m_1},\frac{b_2-a_2}{m_2}} \phi(a_1, a_2 + k_2 \tfrac{b_2-a_2}{m_2}),
\tag{34}
$$

$$
C_{i_2,2}(\phi, \underline{m}) = \tbinom{m_2}{i_2}\tbinom{m_1}{s_1-1} \sum_{k_1=0}^{m_1-s_1} \Delta^{(s_1,i_2)}_{\frac{b_1-a_1}{m_1},\frac{b_2-a_2}{m_2}} \phi(a_1 + k_1 \tfrac{b_1-a_1}{m_1}, a_2),
\tag{35}
$$

$$
\begin{aligned}
C(\phi, \underline{m}) \;=\; & \tbinom{m_1}{s_1-1}\tbinom{m_2}{s_2-1} \Bigg\{ \sum_{k_1=0}^{m_1-s_1} \Delta^{(s_1,s_2-1)}_{\frac{b_1-a_1}{m_1},\frac{b_2-a_2}{m_2}} \phi(a_1 + k_1 \tfrac{b_1-a_1}{m_1}, a_2) \\
& + \sum_{k_2=0}^{m_2-s_2} \Delta^{(s_1-1,s_2)}_{\frac{b_1-a_1}{m_1},\frac{b_2-a_2}{m_2}} \phi(a_1 + (m_1 - s_1 + 1) \tfrac{b_1-a_1}{m_1}, a_2 + k_2 \tfrac{b_2-a_2}{m_2}) \Bigg\}.
\end{aligned}
\tag{36}
$$

Proof. Let us take $[\underline{a},\underline{b}] \equiv [\underline{0},\underline{1}]$. By Lemma 3.2, we know that, since ϕ is a $\underline{s}$-increasing convex function on $[\underline{0},\underline{1}]$, the polynomial $B_{\underline{m}}(\phi;.), \underline{m} \geq \underline{s}$, is regular $\underline{s}$-increasing convex on $[\underline{0},\underline{1}]$. Thus, we may apply Lemma 3.1, and $B_{\underline{m}}(\phi;.)$ can be expressed as

$$(37)\qquad \begin{aligned} B_{\underline{m}}(\phi;\underline{x}) &= \Omega_{\underline{s}-\underline{1}}[B_{\underline{m}}(\phi;.);\underline{x}] + \sum_{i_1=0}^{s_1-2} \frac{C_{i_1,1}(\phi,\underline{m})}{n} \sum_{k=1}^{n-1} x_1^{i_1}(x_2-\alpha_k^{(i_1)})_+^{s_2-1} \\ &\quad + \sum_{i_2=0}^{s_2-2} \frac{C_{i_2,2}(\phi,\underline{m})}{n} \sum_{k=1}^{n-1} x_2^{i_2}(x_1-\beta_k^{(i_2)})_+^{s_1-1} \\ &\quad + \frac{C(\phi,\underline{m})}{n^2} \sum_{\ell=1}^{n-1}\sum_{k=1}^{n-1} (x_1-\epsilon_\ell)_+^{s_1-1}(x_2-\vartheta_k)_+^{s_2-1} \\ &\quad + R_1^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}] + R_2^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}] + R^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}], \end{aligned}$$

for some points $\alpha_k^{(i_1)}, \beta_k^{(i_2)}, \epsilon_\ell, \vartheta_k$ in $[0,1]$, and with coefficients

$$(38)\qquad \begin{aligned} C_{i_1,1}(\phi,\underline{m}) &= \tfrac{1}{i_1!\,(s_2-1)!} \int_{t_2=0}^{1} \{B_{\underline{m}}[\phi;(0,t_2)]\}^{(i_1,s_2)} \mathrm{d}t_2, \\ C_{i_2,2}(\phi,\underline{m}) &= \tfrac{1}{i_2!\,(s_1-1)!} \int_{t_1=0}^{1} \{B_{\underline{m}}[\phi;(t_1,0)]\}^{(s_1,i_2)} \mathrm{d}t_1, \\ C(\phi,\underline{m}) &= \tfrac{1}{(s_1-1)!(s_2-1)!} \bigg\{ \int_{t_2=0}^{1} \{B_{\underline{m}}[\phi;(1,t_2)]\}^{(s_1-1,s_2)} \mathrm{d}t_2 \\ &\qquad + \int_{t_1=0}^{1} \{B_{\underline{m}}[\phi;(t_1,0)]\}^{(s_1,s_2-1)} \mathrm{d}t_1 \bigg\}, \end{aligned}$$

for $i_1 = 0,\ldots,s_1-2$ and $i_2 = 0,\ldots,s_2-2$. Note that these coefficients are all non-negative. Moreover, from (14), (15) and (16), the remainders in (37) are bounded by

$$(39)\qquad \begin{aligned} \big|R_1^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}]\big| &\leq \tfrac{e}{n} \sup_{i_1\leq s_1-2} [i_1! C_{i_1,1}(\phi,\underline{m})] \equiv \frac{C_1(\phi,\underline{m})}{n}, \\ \big|R_2^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}]\big| &\leq \tfrac{e}{n} \sup_{i_2\leq s_2-2} [i_2! C_{i_2,2}(\phi,\underline{m})] \equiv \frac{C_2(\phi,\underline{m})}{n}, \\ \big|R^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}]\big| &\leq \frac{2C(\phi,\underline{m})}{n} \equiv \frac{\tilde{C}(\phi;\underline{m})}{n}. \end{aligned}$$

Now, let us introduce the function $\psi_{\underline{m}}^{(n)}(\phi,.)$ defined as the the right-hand side member of (37) without the remainders, i.e.

$$(40)\qquad \psi_{\underline{m}}^{(n)}(\phi;\underline{x}) \equiv B_{\underline{m}}(\phi;\underline{x}) - \Big\{ R_1^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}] + R_2^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}] + R^{(n)}[B_{\underline{m}}(\phi;.);\underline{x}] \Big\}.$$

From (39) and (40), we have

$$\left|B_{\underline{m}}(\phi;\underline{x}) - \psi^{(n)}_{\underline{m}}(\phi;\underline{x})\right| \le \tfrac{1}{n}\{C_1(\phi,\underline{m}) + C_2(\phi,\underline{m}) + \tilde{C}(\phi,\underline{m})\}.$$

Therefore, we get that, for any $\underline{x} \in [\underline{0},\underline{1}]$,

$$\begin{aligned} \left|\phi(\underline{x}) - \psi^{(n)}_{\underline{m}}(\phi;\underline{x})\right| &\le \\ &\le \left|\phi(\underline{x}) - B_{\underline{m}}(\phi;\underline{x})\right| + \left|B_{\underline{m}}(\phi;\underline{x}) - \psi^{(n)}_{\underline{m}}(\phi;\underline{x})\right| \\ &\le \sup_{\underline{x}\in[\underline{0},\underline{1}]} \left|\phi(\underline{x}) - B_{\underline{m}}(\phi;\underline{x})\right| + \tfrac{1}{n}\{C_1(\phi,\underline{m}) + C_2(\phi,\underline{m}) + \tilde{C}(\phi,\underline{m})\}. \end{aligned} \tag{41}$$

We know that for $\underline{m}$ large enough, the term $\sup|\ldots|$ in (41) is small. Now, let us choose n so large that the other term $(1/n)\{\ldots\}$ in (41) is small. We then see that any function $\phi \in \overline{\mathcal{U}}^{[\underline{0},\underline{1}]}_{\underline{s}-icx}$ can be approximated uniformly on $[\underline{0},\underline{1}]$ by the spline functions $\psi^{(n)}_{\underline{m}}(\phi;.)$. To end with, it suffices to check that $\psi^{(n)}_{\underline{m}}(\phi;.)$ given by (40) with (37), (38) is equivalent to (33) with (34), (35), (36). This is easily shown using the formula (31) for $\{B_{\underline{m}}(\phi;\underline{x})\}^{(\ell_1+1,\ell_2)}$. For instance, from (38), we then get

$$\begin{aligned} C_{i_1,1}(\phi,\underline{m}) = &\tfrac{1}{i_1!\,(s_2-1)!}\tfrac{m_1!}{(m_1-i_1)!}\tfrac{m_2!}{(m_2-s_2)!}\cdot \\ &\cdot \sum_{k_2=0}^{m_2-s_2} \Delta^{(i_1,s_2)}_{\frac{1}{m_1},\frac{1}{m_2}}\phi(0,\tfrac{k_2}{m_2})\tbinom{m_2-s_2}{k_2}\int_{t_2=0}^{1} t_2^{k_2}(1-t_2)^{m_2-s_2-k_2}\mathrm{d}t_2, \end{aligned}$$

and since the latter integral is equal to $1/\{\binom{m_2-s_2}{k_2}(m_2-s_2+1)\}$,

$$C_{i_1,1}(\phi,\underline{m}) = \tbinom{m_1}{i_1}\tbinom{m_2}{s_2-1}\sum_{k_2=0}^{m_2-s_2}\Delta^{(i_1,s_2)}_{\frac{1}{m_1},\frac{1}{m_2}}\phi(0,\tfrac{k_2}{m_2}), \tag{42}$$

such as indicated in (34). □

Remark 3.4. The result of Proposition 3.3 holds for every function $\phi \in \overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$ with $\underline{s} \ge \underline{1}$, that is also if s_1 or $s_2 = 1$. In fact, when $\underline{s} = \underline{1}$, a direct argument yields that ϕ can be approximated uniformly by functions $\phi^{(n)}$ of the form

$$\phi^{(n)}(\underline{x}) = \phi(a_1,a_2) + \frac{\phi(b_1,b_2)-\phi(a_1,a_2)}{n^2}\sum_{\ell=1}^{n-1}\sum_{k=1}^{n-1}(x_1-\epsilon_\ell)^0_+(x_2-\vartheta_k)^0_+,$$

and $\phi^{(n)}$ corresponds precisely to $\psi^{(n)}_{\underline{m}}(\phi;.)$ given in (33), independently of $\underline{m}$ (with the convention that an empty sum is equal to 0). When $s_1 = 1$ and $s_2 = 2$, or inversely, it can be shown, by combining a direct argument and the method of proof above, that (33) provides again a uniform approximation for ϕ. □

Let us return to the original problem of the extremal generators for the bivariate stochastic orderings of increasing convex type. The main result is Corollary 3.6 below which deals with the maximal generator $\overline{\mathcal{F}}$.

Corollary 3.5. *Let* $\underline{X}$ *and* $\underline{Y}$ *be two bivariate random variables valued in* $[\underline{0},\underline{1}]$. *Then, for* $\underline{s} \geq \underline{1}$, $\underline{X} \preceq^{[\underline{a},\underline{b}]}_{\underline{s}-icx} \underline{Y}$ *if and only if* (1) *holds with* $\underline{\mathcal{F}} = \underline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$.

Proof. The sufficiency part follows directly by writing Taylor's expansion of $\phi(.)$ of degree $s_1 - 1, s_2 - 1$ (instead of $s_1 - 2, s_2 - 2$ as with (17)), and then taking the expectations. For the necessity part, it suffices to apply the property that any function $\phi \in \underline{\mathcal{U}}^{[\underline{0},\underline{1}]}_{\underline{s}-icx}$ is the uniform limit of some sequence of functions $\phi^{(n)} \in \mathcal{U}^{[\underline{0},\underline{1}]}_{\underline{s}-icx}$ (see, e.g., Denuit et al. [6], proof of Theorem 3.5). □

Corollary 3.6. *Let* $\underline{X}$ *and* $\underline{Y}$ *be two bivariate random variables valued in* $[\underline{0},\underline{1}]$. *Then, for* $\underline{s} \geq \underline{1}$, $\underline{X} \preceq^{[\underline{a},\underline{b}]}_{\underline{s}-icx} \underline{Y}$ *if and only if* (1) *holds with* $\overline{\mathcal{F}} = \overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$.

Proof. The sufficiency part is immediate from (9). For the necessity part, let $\phi \in \overline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$. By Proposition 3.3, ϕ is the uniform limit of some sequence of functions $\phi^{(n)}$, implying that $E\phi^{(n)}(X) \to E\phi(X)$ and $E\phi^{(n)}(Y) \to E\phi(Y)$ as $n \to +\infty$. Moreover, these $\phi^{(n)}$'s are non-negative linear combinations of functions in $\underline{\mathcal{U}}^{[\underline{a},\underline{b}]}_{\underline{s}-icx}$, so that by Corollary 3.5, $E\phi^{(n)}(X) \leq E\phi^{(n)}(Y)$ for all n. Therefore, we deduce that $E\phi(X) \leq E\phi(Y)$. □

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