REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 32 (2003) no. 2, pp. 161–169
ictp.acad.ro/jnaat

AN OSTROWSKI TYPE INEQUALITY FOR DOUBLE INTEGRALS IN TERMS OF L_p-NORMS AND APPLICATIONS IN NUMERICAL INTEGRATION

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Abstract. An inequality of the Ostrowski type for double integrals and applications in Numerical Analysis in connection with cubature formulae are given.

MSC 2000. Primary: 26D15; Secondary: 41A55.

Keywords. Ostrowski's inequality, cubature formulae.

1. INTRODUCTION

In 1938, A. Ostrowski proved the following integral inequality [5, p. 468].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e.,*

$$\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty.$$

Then we have the inequality

$$\left|f(x) - \tfrac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leq \left[\tfrac{1}{4} + \tfrac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \quad \forall x\in[a,b].$$

The constant $\frac{1}{4}$ is the best possible.

For some generalizations see the book [5, pp. 468–484] by Mitrinović, Pečarić and Fink.

Some applications of the above results in Numerical Integration and for special means have been given in [3] by S. S. Dragomir and S. Wang.

In [4] Dragomir and Wang established the following Ostrowski type inequality for differentiable mappings whose derivatives belong to L_p-spaces.

Theorem 2. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $f'\in L_p(a,b)$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then we have the inequality:*

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$$\left|f(x)-\tfrac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\le\tfrac{1}{b-a}\left[\tfrac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|f'\right\|_p,\quad\forall x\in[a,b],$$

where

$$\left\|f'\right\|_p:=\left(\int_a^b\left|f'(t)\right|^p\mathrm{d}t\right)^{\frac{1}{p}},$$

is the $L_p(a,b)$-norm.

Note that the above inequality can also be obtained from Theorem 1 [5, p. 471] due to A. M. Fink.

For other Ostrowski type inequalities, see the papers [1], [2] and [4].

In 1975, G. N. Milovanović generalized Theorem 1, where f is a function of several variables [5, p. 468].

THEOREM 3. *Let $f:\mathbb{R}^m\to\mathbb{R}$ be a differentiable function defined on $D=\{(x_1,\dots,x_m): a_i\le x_i\le b_i,\ i=1,\dots,m\}$ and let $\left|\frac{\partial f}{\partial x_i}\right|\le M_i$, $M_i>0$, $i=1,\dots,m$, in D. Furthermore, let function $x\longmapsto p(x)$ be integrable and $p(x)>0$, for every $x\in D$. Then for every $x\in D$, we have the inequality:*

$$\left|f(x)-\frac{\int_D p(y)f(y)\,\mathrm{d}y}{\int_D p(y)\,\mathrm{d}y}\right|\le\frac{\sum_{i=1}^m M_i\int_D p(y)\left|x_i-y_i\right|\mathrm{d}y}{\int_D p(y)\,\mathrm{d}y}.$$

In the present paper we point out an Ostrowski type inequality for double integrals in terms of L_p-norms and apply it in Numerical Integration obtaining a general cubature formula.

2. THE RESULTS

The following inequality of Ostrowski's type for mappings of two variables holds:

THEOREM 4. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous mapping on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is in $L_p((a,b)\times(c,d))$, i.e.,*

$$\left\|f''_{s,t}\right\|_p:=\left(\int_a^b\int_c^d\left|\tfrac{\partial^2 f(x,y)}{\partial x\partial y}\right|^p\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}<\infty,\quad p>1,$$

then we have the inequality:

$$\begin{aligned}(1)\quad &\left|\int_a^b\int_c^d f(s,t)\,\mathrm{d}s\mathrm{d}t-\Big[(b-a)\int_c^d f(x,t)\,\mathrm{d}t+(d-c)\int_a^b f(s,y)\,\mathrm{d}s\right.\\ &\left.\quad-(d-c)(b-a)f(x,y)\Big]\right|\le\end{aligned}$$

$$\leq \left[\tfrac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left[\tfrac{(y-c)^{q+1}+(d-y)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left\| f''_{s,t} \right\|_p$$

for all $(x,y) \in [a,b] \times [c,d]$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Proof. Integrating by parts successively, we have the equality:

$$\begin{aligned}(2)\quad & \int_a^x \int_c^y (s-a)(t-c) f''_{s,t}(s,t)\,\mathrm{d}t\mathrm{d}s = \\ & = (y-c)(x-a) f(x,y) - (y-c)\int_a^x f(s,y)\,\mathrm{d}s - (x-a)\int_c^y f(x,t)\,\mathrm{d}t \\ & \quad + \int_a^x \int_c^y f(s,t)\,\mathrm{d}s\mathrm{d}t.\end{aligned}$$

By similar computations, we have

$$\begin{aligned}(3)\qquad & \int_a^x \int_y^d (s-a)(t-d) f''_{s,t}(s,t)\,\mathrm{d}s\mathrm{d}t \\ & = (x-a)(d-y) f(x,y) - (d-y)\int_a^x f(s,y)\,\mathrm{d}s \\ & \quad - (x-a)\int_y^d f(x,t)\,\mathrm{d}t + \int_a^x \int_c^y f(s,t)\,\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Now,

$$\begin{aligned}(4)\qquad & \int_x^b \int_y^d (s-b)(t-d) f''_{s,t}(s,t)\,\mathrm{d}s\mathrm{d}t \\ & = (d-y)(b-x) f(x,y) - (d-y)\int_x^b f(s,y)\,\mathrm{d}s \\ & \quad - (b-x)\int_y^d f(x,t)\,\mathrm{d}t + \int_x^b \int_y^d f(s,t)\,\mathrm{d}s\mathrm{d}t\end{aligned}$$

and finally

$$\begin{aligned}(5)\qquad & \int_x^b \int_c^y (s-b)(t-c) f''_{s,t}(s,t)\,\mathrm{d}s\mathrm{d}t \\ & = (y-c)(b-x) f(x,y) - (y-c)\int_x^b f(s,y)\,\mathrm{d}s \\ & \quad - (b-x)\int_c^y f(x,t)\,\mathrm{d}t + \int_x^b \int_c^y f(s,t)\,\mathrm{d}s\mathrm{d}t.\end{aligned}$$

If we add the equalities (2) – (5) we get, in the right hand side:

$$\Big[(y-c)(x-a) + (x-a)(d-y) + (d-y)(b-x) + (y-c)(b-x)\Big] f(x,y) -$$

$$
\begin{aligned}
& -(d-c)\int_a^x f(s,y)\,\mathrm{d}s-(d-c)\int_x^b f(s,y)\,\mathrm{d}s-(b-a)\int_c^y f(x,t)\,\mathrm{d}t \\
& -(b-a)\int_y^d f(x,t)\,\mathrm{d}t+\int_a^x\int_c^y f(s,t)\,\mathrm{d}s\mathrm{d}t+\int_a^x\int_y^d f(s,t)\,\mathrm{d}s\mathrm{d}t \\
& +\int_x^b\int_y^d f(s,t)\,\mathrm{d}s\mathrm{d}t+\int_x^b\int_c^y f(s,t)\,\mathrm{d}s\mathrm{d}t \\
= & (d-c)(b-a)f(x,y)-(d-c)\int_a^b f(s,y)\,\mathrm{d}s-(b-a)\int_c^b f(x,t)\,\mathrm{d}t \\
& +\int_a^b\int_c^d f(s,t)\,\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

For the first part, let us define the kernels: $p:[a,b]^2\to\mathbb{R}$, $q:[c,d]^2\to\mathbb{R}$ given by:

$$
p(x,s):=\begin{cases} s-a, & \text{if } s\in[a,x], \\ s-b, & \text{if } s\in(x,b] \end{cases}
$$

and

$$
q(y,t):=\begin{cases} t-c, & \text{if } t\in[c,y], \\ t-d, & \text{if } t\in(y,d]. \end{cases}
$$

Now, we deduce that the left part can be represented as:

$$
\int_a^b\int_c^d p(x,s)\,q(y,t)\,f''_{s,t}(s,t)\,\mathrm{d}s\mathrm{d}t.
$$

Consequently, we get the identity

$$
\begin{aligned}
(6)\qquad & \int_a^b\int_c^d p(x,s)\,q(y,t)\,f''_{s,t}(s,t)\,\mathrm{d}s\mathrm{d}t= \\
& =(d-c)(b-a)f(x,y)-(d-c)\int_a^b f(s,y)\,\mathrm{d}s \\
& \quad -(b-a)\int_c^d f(x,t)\,\mathrm{d}t+\int_a^b\int_c^d f(s,t)\,\mathrm{d}s\mathrm{d}t,
\end{aligned}
$$

for all $(x,y)\in[a,b]\times[c,d]$.

Now, using the identity (6), we get

$$
\begin{aligned}
& \Big|\int_a^b\int_c^d f(s,t)\mathrm{d}s\mathrm{d}t-\Big[(b-a)\int_c^d f(x,t)\mathrm{d}t+(d-c)\int_a^b f(s,y)\mathrm{d}s \\
& \quad -(d-c)(b-a)f(x,y)\Big]\Big|\le \\
& \le\int_a^b\int_c^d |p(x,s)q(y,t)|\cdot|f''_{s,t}(s,t)|\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Using Hölder's integral inequality for double integrals, we get

$$\begin{aligned}
&\int_a^b\int_c^d |p(x,s)\,q(y,t)|\left|f''_{s,t}(s,t)\right|\mathrm{d}s\mathrm{d}t \leq \\
&\leq \left(\int_a^b\int_c^d |p(x,s)\,q(y,t)|^q\,\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_a^b\int_c^d \left|f''_{s,t}(s,t)\right|^p\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}} \\
&= \left(\int_a^b |p(x,s)|^q\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_c^d |q(y,t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left\|f''_{s,t}\right\|_p \\
&= \left[\tfrac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left[\tfrac{(y-c)^{q+1}+(d-y)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|f''_{s,t}\right\|_p
\end{aligned}$$

and the theorem is proved. □

Corollary 5. *Under the above assumptions, we have the inequality:*

$$\begin{aligned}
(7)\qquad &\left|\int_a^b\int_c^d f(s,t)\,\mathrm{d}s\mathrm{d}t - \left[(b-a)\int_c^d f\left(\tfrac{a+b}{2},t\right)\mathrm{d}t\right.\right. \\
&\left.\left. + (d-c)\int_a^b f\left(s,\tfrac{c+d}{2}\right)\mathrm{d}s - (d-c)(b-a)\,f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right)\right]\right| \leq \\
&\leq \tfrac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\left\|f''_{s,t}\right\|_p .
\end{aligned}$$

Remark 1. Consider the mapping $g:[\alpha,\beta]\to\mathbb{R}, g(t)=(t-\alpha)^m+(\beta-t)^m$, $m\geq 1$. Taking into account the fact that one has the properties

$$\inf_{t\in[\alpha,\beta]} g(t) = g\left(\tfrac{\alpha+\beta}{2}\right) = \tfrac{(\beta-\alpha)^m}{2^{m-1}}$$

and

$$\sup_{t\in[\alpha,\beta]} g(t) = g(\alpha) = g(\beta) = (\beta-\alpha)^m$$

then, the above inequality (7) is the best that can be obtained from (1). □

Remark 2. Now, if we assume that $f(s,t) = h(s)\,h(t)$, $h:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and suppose that $\|h'\|_p<\infty$, then from (1) we get, for $x=y$,

$$\begin{aligned}
&\left|\int_a^b h(s)\,\mathrm{d}s\int_a^b h(s)\,\mathrm{d}s - h(x)(b-a)\int_a^b h(s)\,\mathrm{d}s\right. \\
&\left. -h(x)(b-a)\int_a^b h(s)\,\mathrm{d}s + (b-a)^2 h^2(x)\right| \leq \\
&\leq \left[\tfrac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{2}{q}}\|h'\|_p^2,
\end{aligned}$$

i.e.,

$$\left[\int_a^b h(s)\,\mathrm{d}s - h(x)(b-a)\right]^2 \leq \left[\tfrac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{2}{q}} \|h'\|_p^2,$$

which is clearly equivalent to Ostrowski's inequality. Consequently (1) can be also regarded as a generalization for double integrals of the result embodied in Theorem 2. □

3. APPLICATIONS FOR CUBATURE FORMULAE

Let us consider the arbitrary divisions I_n: $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, J_m: $c = y_0 < y_1 < \ldots < y_{m-1} < y_m = b$ and $\xi_i \in [x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, $\eta_j \in [y_j, y_{j+1}]$, $j = 0, \ldots, m-1$, be intermediate points. Consider the sum

$$C(f, I_n, J_m, \xi, \eta) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\,\mathrm{d}t + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\,\mathrm{d}s - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j),$$

for which we assume that the involved integrals can more easily be computed than the original double integral

$$D := \int_a^b\int_c^d f(s,t)\,\mathrm{d}s\mathrm{d}t,$$

and

$$h_i := x_{i+1} - x_i, \quad i = 0, \ldots, n-1, \quad l_j := y_{j+1} - y_j, \quad j = 0, \ldots, m-1.$$

With this assumption, we can state the following cubature formula:

Theorem 6. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 4 and I_n, J_m, ξ and η be as above. Then we have the cubature formula:*

$$\int_a^b\int_c^d f(s,t)\,\mathrm{d}s\mathrm{d}t = C(f, I_n, J_m, \xi, \eta) + R(f, I_n, J_m, \xi, \eta)$$

where the remainder term $R(f, I_n, J_m, \xi, \eta)$ satisfies the estimation:

$$\begin{aligned}(8)\quad |R(f, I_n, J_m, \xi, \eta)| &\leq \\ &\leq \left\|f''_{s,t}\right\|_p \left[\sum_{i=0}^{n-1} \tfrac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1}\right]^{\frac{1}{q}} \left[\sum_{j=0}^{m-1} \tfrac{(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}}{q+1}\right]^{\frac{1}{q}} \\ &\leq \frac{\|f''_{s,t}\|_p}{(q+1)^{\frac{2}{q}}} \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}},\end{aligned}$$

for all ξ and η as above.

Proof. Apply Theorem 4 on the interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0, \ldots, n-1; j = 0, \ldots, m-1$, to get:

$$\left| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(s,t)\mathrm{d}s\mathrm{d}t - \left[h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\,\mathrm{d}t + l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\,\mathrm{d}s - h_i l_j f(\xi_i, \eta_j) \right] \right|$$

$$\leq \left[\frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1} \cdot \frac{(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left[\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |\, f(s,t)\,|^p \,\mathrm{d}s\mathrm{d}t \right]^{\frac{1}{p}}$$

for all $i = 0, \ldots, n-1; j = 0, \ldots, m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality and Hölder's discrete inequality for double sums, we deduce

$$\begin{aligned}
&|R(f, I_n, J_m, \xi, \eta)| \leq \\
&\leq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1} \cdot \frac{(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}}{q+1} \right]^{\frac{1}{q}} \\
&\qquad \times \left[\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |f(s,t)|^p \mathrm{d}s\mathrm{d}t \right]^{\frac{1}{p}} \\
&\leq \left[\sum_{i=0}^{n-1} \left(\frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}} \left[\sum_{j=0}^{m-1} \left(\frac{(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}} \\
&\quad \times \left[\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |f(s,t)|^p \mathrm{d}s\mathrm{d}t \right]^{\frac{1}{p}} \\
&= \left[\sum_{i=0}^{n-1} \frac{(x_{i+1}-\xi_i)^{q+1}+(\xi_i-x_i)^{q+1}}{q+1} \times \sum_{j=0}^{m-1} \frac{(y_{j+1}-\eta_j)^{q+1}+(\eta_j-y_j)^{q+1}}{q+1} \right]^{\frac{1}{q}} \times \left\| f''_{s,t} \right\|_p .
\end{aligned}$$

To prove the second part, we observe that

$$(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1} \leq (x_{i+1} - x_i)^{q+1}$$

and

$$(y_{j+1} - \eta_j)^{q+1} + (\eta_j - y_j)^{q+1} \leq (y_{j+1} - y_j)^{q+1},$$

for all i, j as above and the intermediate points ξ_i and η_j.

We omit the details. □

Remark 3. As

$$\sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \leq [\nu(h)]^{\frac{1}{q}} \sum_{i=0}^{n-1} h_i = (b-a)\,[\nu(h)]^{\frac{1}{q}}$$

and

$$\sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}} \leq [\mu(l)]^{\frac{1}{q}} \sum_{j=0}^{m-1} l_j = (d-c)[\mu(l)]^{\frac{1}{q}},$$

where

$$\nu(h) = \max\{h_i : i = 0, \ldots, n-1\}$$

and

$$\mu(l) = \max\{l_j : j = 0, \ldots, m-1\},$$

the right hand side of (8) can be bounded by

$$\frac{1}{(q+1)^{2/q}} \left\| f''_{s,t} \right\|_p (b-a)(d-c)[\nu(h)\mu(l)]^{\frac{1}{q}}. \qquad \square$$

Now, define the sum

$$\begin{aligned} C_M(f, I_n, J_m) := & \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f\left(\tfrac{x_i + x_{i+1}}{2}, t\right) \mathrm{d}t \\ & + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f\left(s, \tfrac{y_j + y_{j+1}}{2}\right) \mathrm{d}s \\ & - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f\left(\tfrac{x_i + x_{i+1}}{2}, \tfrac{y_j + y_{j+1}}{2}\right). \end{aligned}$$

Then we have the best cubature formula we can get from Theorem 6.

Corollary 7. *Under the above assumptions we have*

$$\int_a^b \int_c^d f(s,t)\, \mathrm{d}s\mathrm{d}t = C_M(f, I_n, J_m) + R(f, I_n, J_m),$$

where the remainder $R(f, I_n, J_m)$ satisfies the estimation:

$$|R(f, I_n, J_m)| \leq \frac{1}{4(q+1)^{2/q}} \left\| f''_{s,t} \right\|_p \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}}.$$

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Received by the editors: September 29, 1998.