REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
**Rev. Anal. Numér. Théor. Approx., vol. 32 (2003) no. 2, pp. 187–192**
ictp.acad.ro/jnaat

# ON THE UNIQUENESS OF EXTENSION AND UNIQUE BEST APPROXIMATION IN THE DUAL OF AN ASYMMETRIC NORMED LINEAR SPACE

COSTICĂ MUSTĂŢA*

**Abstract.** A well known result of R. R. Phelps (1960) asserts that in order that every linear continuous functional, defined on a subspace $Y$ of a real normed space $X$, have a unique norm preserving extension it is necessary and sufficient that its annihilator $Y^{\perp}$ be a Chebyshevian subspace of $X^*$. The aim of this note is to show that this result holds also in the case of spaces with asymmetric norm.

**MSC 2000.** 41A65, 46A22, 46B20.

**Keywords.** Asymmetric normed spaces, extensions preserving asymmetric norm, best approximation.

## 1. INTRODUCTION

Let $X$ be a real linear space. A function $\|\cdot| : X \to [0, \infty)$ is called an *asymmetric norm* if it satisfies all the usual axioms of a norm, excepting the absolute homogeneity, which is replaced by positive homogeneity, i.e.,

$$\|\lambda x| = \lambda \|x|, \quad \forall x \in X, \ \forall \lambda \geq 0.$$

The asymmetric norm is called with *extended values* if there exists $x \in X$ such that $\|x| = +\infty$. The pair $(X, \|\cdot|)$ is called a *space with asymmetric norm* (see [8], [2]).

In a space with asymmetric norm it is possible that $\|x| \neq \|-x|$ for some $x \in X$. The asymmetric norm generates a topology having as a neighborhood base the balls $B(x,r) = \{y \in X : \|y - x| < r\}$, $x \in X$, $r \geq 0$, but the topological space $\left(X, \tau_{\|\cdot|}\right)$ is not a linear topological space, because the multiplication by scalars is not a continuous operation (see [2, p. 199]).

Let $X^{\#}$ be the algebraic dual of $X$, i.e. the space of all linear functional on $X$. We say that $f \in X^{\#}$ is *bounded* on $X$ if

$$\sup\left\{f(x) : x \in X, \ \|x| \leq 1\right\} < \infty. \tag{1}$$

It is immediate that if $f, g \in X^{\#}$ are bounded then their sum $f + g$ and the product $\lambda f$ for $\lambda \geq 0$ are bounded too. This shows that the set of all bounded linear functionals on $X$ is a cone (see [2]), or an ac-space, according to [7].

*"T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68-1, 3400 Cluj-Napoca, Romania, e-mail: cmustata@ictp.acad.ro, cmustata2001@yahoo.com.

In general, it is possible that for a bounded linear functional $f$ on $(X, \|\cdot|)$ the linear functional $-f$ be not bounded. Such an example is given by the functional $f(x) = x(1)$ defined on the space

$$X = \Big\{x : [0,1] \to \mathbb{R} \mid x \text{ continuous and } \int_0^1 x(t)\,\mathrm{d}t = 0\Big\}$$

equipped with the asymmetric norm

$$\|x| = \max\{x(t) : t \in [0,1]\},\ x \in X.$$

Taking $x_n(t) = 1 - nt^{n-1}$ it follows $\|x_n| = 1$ and $(-f)(x_n) = f(-x_n) = (-x_n)(1) = n-1$, implying

$$\sup\{(-f)(x) : \|x| \le 1\} \ge \sup\{(-f)(x_n) : n \in \mathbb{N}\} = +\infty.$$

For a bounded linear functional $f$ put

$$\|f| = \sup\{f(x) : x \in X, \|x| \le 1\}. \tag{2}$$

It follows that the function $\|\cdot|$ defined by (2) satisfies the axioms of an asymmetric norm on the cone of all bounded linear functionals on $X$ (see [2]).

Observe that

$$f(x) \le \|f| \cdot \|x|, \quad x \in X,$$

and, since $f(-x) \le \|f| \cdot \|-x|$, one obtains the inequalities

$$-\|f| \cdot \|-x| \le f(x) \le \|f| \cdot \|x|, \quad x \in X.$$

The inequality $|f(x)| \le \|f| \cdot \|x|$ is not true in general, but if we consider the symmetric norm $\|x\| = \max\{\|x|, \|-x|\}$ on $X$, then $|f(x)| \le \|f|\,\|x\|$, $x \in X$. This shows that a bounded linear functional is always continuous with respect to the topology generated by the symmetric norm $\|x\| = \max\{\|x|, \|-x|\}$ associated to an asymmetric norm $\|\cdot|$.

If both $f$ and $-f$ are bounded then the linear functional $f$ is continuous with respect to the topology generated by the asymmetric norm.

Consider on $\mathbb{R}$ the asymmetric norm $u(x) = x \vee 0 = \max\{x, 0\}$ (see [7]). A functional $f : (X, \|\cdot|) \to (\mathbb{R}, u)$ is continuous in $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ with $\|x - x_0| < \delta$ we have $(f(x) - f(x_0)) \vee 0 < \varepsilon$.

It is clear that a linear functional $f : (X, \|\cdot|) \to (\mathbb{R}, u)$ is continuous if and only if there exists $M > 0$ such that $f(x) \vee 0 < M\,\|x|$.

According to [7], the set

$$X^* = \{f : (X, \|\cdot|) \to (\mathbb{R}, u),\ f \text{ linear continuous}\} \tag{3}$$

is called the *asymmetric dual* of the space with asymmetric norm $(X, \|\cdot|)$.

If $\|x\| = \max\{\|x|, \|-x|\}$ is the norm generated by $\|\cdot|$ on $X$ and $\mathbb{R}$ is equipped with the usual absolute-value norm $|\cdot|$, then the set

$$X^{*s} = \{f : (X, \|\cdot\|) \to (\mathbb{R}, |\cdot|),\ f \text{ linear continuous}\} \tag{4}$$

is the (symmetric) dual of $(X, \|\cdot|)$. In other words, $X^{*s}$ is the usual topological algebraic dual of the normed space $(X, \|\cdot\|)$, where $\|x\| = \max\{\|x|, \|-x|\}$.

Observe that $X^*$ is a cone in the linear space $X^{*s}$.

Equip the linear space $X^{*s}$ with the extended asymmetric norm

$$\|f|^{*s} = \sup\{f(x) : \|x| \leq 1\}$$

whose restriction to the asymmetric dual $X^*$ is

$$\|f|^* = \sup\{f(x) \vee 0 : \|x| \leq 1\}.$$

It is important to remark the fact that a linear functional $f$ belongs to $X^*$ if and only if it is an upper semicontinuous linear functional on $(X, \|\cdot|)$, and that $X^* = \{f \in X^{*s} : \|f|^{*s} < +\infty\}$ (see [7]).

Let $(Y, \|\cdot|)$ be a subspace of the space with asymmetric norm $(X, \|\cdot|)$, and let $Y^*$ and $Y^{*s}$ be the dual cone and the (symmetric) dual of $Y$, respectively.

The following Hahn–Banach type theorem holds:

THEOREM 1. *Let $(X, \|\cdot|)$ be a real space with asymmetric norm and $(Y, \|\cdot|)$ a subspace of it. Then for every $f \in Y^*$ there exists $F \in X^*$ such that*

$$F|_Y = f \quad \text{and} \quad \|F|^* = \|f|^*.$$

*Proof.* For $f \in Y^*$ let $p : X \to \mathbb{R}$ be defined by $p(x) = \|f|^* \cdot \|x|$, $x \in X$.

The functional $p$ is convex, positively homogeneous and $f(y) \leq \|f|^* \cdot \|y|$, for every $y \in Y$. By the Hahn–Banach extension theorem ([8, p. 484]) there exists a linear functional $F : X \to \mathbb{R}$ such that $F|_Y = f$ and $F(x) \leq \|f|^* \|x|$, for every $x \in X$. It follows

$$\|F|^* = \sup\{F(x) \vee 0 : \|x| \leq 1\} \leq \|f|^*.$$

On the other hand

$$\begin{aligned} \|F|^* &= \sup\{F(x) \vee 0 : \|x| \leq 1,\ x \in X\} \\ &\geq \sup\{F(y) \vee 0 : \|y| \leq 1,\ y \in Y\} \\ &= \sup\{f(y) \vee 0 : \|y| \leq 1,\ y \in Y\} \\ &= \|f|^*, \end{aligned}$$

showing that $\|F|^* = \|f|^*$. □

For $f \in Y^*$ denote by

$$\mathcal{E}(f) = \{F \in X^* : F|_Y = f \text{ and } \|F|^* = \|f|^*\}$$

the set of all extensions that preserve the asymmetric norm.

By Theorem 1, the set $\mathcal{E}(f)$ is always nonempty.

The problem of finding necessary or/and sufficient conditions in order that every $f \in Y^*$ have a unique norm preserving extension is closely related to a best approximation problem in the space $X^{*s}$ equipped with the asymmetric norm $\|F|^{*s} = \sup\{F(x) : \|x| \leq 1\}$.

Concerning the following notions, in the case of usual spaces, one can consult Singer's book [16].

Let $(Y, \|\cdot|)$ be subspace of $(X, \|\cdot|)$ and let

$$Y^{\perp} = \{G \in X^{*s} : G|_Y = 0\}$$

the annihilator of $Y$ in the space $(X^{*s}, \|\cdot|^{*s})$.

For $F \in X^{*s}$, an element $G_0 \in Y^{\perp}$ is called a best approximation element for $F$ in $Y^{\perp}$ if

$$\|F - G_0|^{*s} = \inf\{\, \|F - g|^{*s} : G \in Y^{\perp}\} = d(F, Y^{\perp}).$$

The set of all best approximation elements for $F$ in $Y^{\perp}$ is denoted by $P_{Y^{\perp}}(F)$. If $P_{Y^{\perp}}(F) \neq \emptyset$ for every $F \in X^{*s}$ one says that $Y^{\perp}$ is *proximinal*, and if $card P_{Y^{\perp}}(F) = 1$ for every $F \in X^{*s}$, then one says that $Y^{\perp}$ is *Chebyshevian.*

THEOREM 2. *Let $(X, \|\cdot|)$ be a space with asymmetric norm and $(Y, \|\cdot|)$ a subspace of it. Then*

a) *The annihilator $Y^{\perp}$ of $Y$ is a proximinal subspace of $X^{*s}$ and, for every $F \in X^{*}$ we have*

$$d(F, Y^{\perp}) = \|F|_Y|^{*}.$$

b) *An element $G \in Y^{\perp}$ is in $P_{Y^{\perp}}(F)$ if and only if $G = F - H$ for some $H \in \mathcal{E}(F|_Y)$, i.e.,*

$$P_{Y^{\perp}}(F) = F - \mathcal{E}(F|_Y).$$

c) *The subspace $Y^{\perp}$ is Chebyshevian in $X^{*s}$ if and only if every functional $f \in Y^{*}$ has a unique norm preserving extension in $X^{*s}$.*

*Proof.* a) Let $F \in X^{*}$. Then $F|_Y \in Y^{*}$ and, by Theorem 1, $\mathcal{E}(F|_Y) \neq \emptyset$. If $H \in \mathcal{E}(F|_Y)$ then

$$F|_Y = H|_Y \quad \text{and} \quad \|F|_Y|^{*} = \|H|^{*},$$

so that $F - H \in Y^{\perp}$. We have

$$\|F|_Y|^{*} = \|H|^{*} = \|H|^{*s} = \|F - (F - H)|^{*s} \geq d(F, Y^{\perp})$$

and, for any $G \in Y^{\perp}$,

$$\|F|_Y|^{*} = \|F|_Y - G|_Y|^{*} \leq \|F - G|^{*s}.$$

Taking the infimum with respect to $G \in Y^{\perp}$ we get

$$\|F|_Y|^{*} \leq d(F, Y^{\perp}).$$

It follows that $d\left(F, Y^{\perp}\right) = \|F|_Y|^{*}$.

b) If $G \in P_{Y^{\perp}}(F)$ then

$$\|F - G|^{*s} = \|F|_Y|^{*}$$

and

$$(F-G)\,|_Y = F|_Y,$$

which shows that $F-G \in \mathcal{E}\,(F|_Y)$. The conclusion holds with $H = F-G$.

c) Follows from b). □

REMARK. 1° Let $X^*$ be the usual topological algebraic dual of the normed space $(X, \|\cdot\|)$, and $Y^*$ the topological algebraic dual of $(Y, \|\cdot\|)$, where $Y$ is a subspace of $X$. R. R. Phelps [14] showed that $Y^\perp$ (the annihilator of $Y$ in $X^*$) is Chebyshevian if and only if every $f \in Y^*$ has a unique norm-preserving extension $F \in X^*$.

2° Some Phelps type duality results and applications can be found in [3], and in the bibliography quoted there. □

## REFERENCES

[1] Alegre, C., Ferrer, J. and Gregori, V., *On the Hahn–Banach theorem in certain linear quasi-uniform structures,* Acta Math. Hungar, **82**, pp. 315–320, 1999.

[2] Borodin, P. A., *The Banach–Mazur Theorem for spaces with asymmetric norm and its applications in convex analysis*, Mathematical Notes, **69**, no. 3, pp. 298–305, 2001.

[3] Cobzaş, S., *Phelps type duality results in best approximation,* Rev. Anal. Numér. Théor. Approx., **31**, no. 1, pp. 29–43, 2002.

[4] Dolzhenko, E. P. and Sevast'yanov, E. A., *Approximation with sign-sensitive weights*, Izv. Ross. Akad. Nauk Ser. Mat. [Russian Acad. Sci. Izv. Math.], **62**, no. 6, pp. 59–102, 1998 and **63**, no. 3, pp. 77–118, 1999.

[5] Ferrer, J., Gregori, V. and Alegre, C., *Quasi-uniform structures in linear lattices*, Rocky Mountain J. Math., **23**, pp. 877–884, 1993.

[6] Garćia-Raffi, L. M., Romaguera, S. and Sanchez Pérez, E. A., *Extension of asymmetric norms to linear spaces*, Rend. Istit. Mat. Trieste, **XXXIII**, pp. 113–125, 2001.

[7] Garćia-Raffi, L. M., Romaguera S. and Sánchez-Pérez, E. A., *The dual space of an asymmetric normed linear space,* Quaestiones Mathematicae, **26**, pp. 83–96, 2003.

[8] Krein, M. G. and Nudel'man, A. A., *The Markov Moment Problem and Extremum Problems*, Nauka, Moscow, 1973 (in Russian).

[9] Kopperman, R. D., *All topologies come from generalized metrics*, Amer. Math. Monthly, **95**, pp. 89–97, 1988.

[10] McShane, E. J., *Extension of range of functions*, Bull. Amer. Math. Soc., **40**, pp. 847–842, 1934.

[11] Mustăţa, C., *Extensions of semi-Lipschitz functions on quasi-metric spaces*, Rev. Anal. Numér. Théor. Approx., **30**, no. 1, pp. 61–67, 2001.

[12] Mustăţa, C., *Extensions of convex semi-Lipschitz functions on quasi-metric linear spaces*, Séminaire de la Théorie de la Meileure Approximation Convexité et Optimization, Cluj-Napoca, le 29 novembre, pp. 85–92, 2001.

[13] Mustăţa, C., *A Phelps type theorem for spaces with asymmetric norms*, Bul. Ştiinţ. Univ. Baia-Mare, Ser. B, **XVIII**, no. 2, pp. 275–280, 2002.

[14] Phelps, R. R., *Uniqueness of Hahn–Banach extension and unique best approximation*, Trans. Amer. Math. Soc., **95**, pp. 238–255, 1960.

[15] Romaguera, S. and Sanchis, M., *Semi-Lipschitz functions and best approximation in quasi-metric spaces*, J. Approx. Theory, **103**, pp. 292–301, 2000.

[16] Singer, I., *Best approximation in normed linear spaces by elements of linear subspaces*, Ed. Acad. RSR, Bucharest, 1967 (in Romanian); English Translation, Springer, Berlin, 1970.

Received by the editors: February 3, 2003.