REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
**Rev. Anal. Numér. Théor. Approx., vol. 32 (2003) no. 2, pp. 203–207**
ictp.acad.ro/jnaat

# ON THE FLOW OF A THIN LIQUID LAYER IN AN INCLINED CHANNEL OF TRIANGULAR TRANSVERSE SECTION DRIVEN BY A SURFACE TENSION GRADIENT. NUMERICAL AND ASYMPTOTIC ANALYSIS

TITUS PETRILA* and EMILIA BORŞA†

**Abstract.** In this paper we investigate the locally unidirectional flow of a thin liquid layer confined to an inclined channel driven simultaneously by a surface tension gradient.

**MSC 2000.** 76D08, 76D45, 76M55.

**Keywords.** Thin film approximation, surface tension gradient.

## 1. INTRODUCTION

We consider the steady behavior of a thin layer (trickle, rivulet). We follow the lubrication approximation to obtain an analytical solution for a rivulet of Newtonian fluid moving down a slowly varying substrate. We study the special case when the transversal sectional profile of the rivulet is triangular and we include the effects of a surface tension gradient. We are interested in the study of the Marangoni effect. We take into account a non-zero tangential stress boundary condition. The extra terms introduced by this non-zero shear stress have a reasonable physical significance.

## 2. PROBLEM FORMULATION

We choose Cartesian system $Oxyz$, with the $x$ axis in the direction of flow and the $y$ axis transverse to the direction of flow and so the fluid velocity is given by $\overrightarrow{u} = u(y,z)\overrightarrow{i}$.

Here (L): $z = (H+h)(y)$ denotes the unknown transverse profile of the free surface, (S): $z = H(y)$ denotes the known transverse profile of the substrate and $h$ is thickness of the layer.

We consider the unidirectional flow of a thin layer of a constant width $a$ of Newtonian fluid with constant density $\rho$ and constant viscosity $\mu$ down a substrate, which is inclined at an angle $\varphi$, $0 \leq \varphi \leq \pi/2$, to the horizontal.

*"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, str. M. Kogălniceanu 1, 3400 Cluj-Napoca, Romania, e-mail: tpetrila@cs.ubbcluj.ro.

†Department of Mathematics, University of Oradea, Romania, e-mail: borsavasile@yahoo.com.

Employing the familiar lubrication approximation [1], [7] the governing equations for this unidirectional flow are simply

$$\begin{aligned}-\mu u_{zz} &= \rho g\ \sin\varphi - p_x,\\ p_y &= 0,\\ p_z &= -\rho g\ \cos\varphi.\end{aligned}$$

They are to be integrated subject to the boundary conditions

$$\begin{aligned}u &= 0, \quad \text{on } z = H(y)\\ u_x &= \tfrac{1}{\mu}\Gamma_x \text{ and } p - p_0 = -\bar{\Gamma} h_{yy}, \quad \text{on } z = (H+h)(y)\\ h &= 0 \text{ and } h_y = \mp\beta,\ \text{on } y = \pm a.\end{aligned}$$

Here $p = p(y,z)$ denotes the pressure in the fluid, $p_0$ is the constant pressure in the ambient gas above the fluid, $\bar{\Gamma}$ is the references value of surface tension, $g$ is the acceleration due to gravity, $\Gamma_x$ is the surface tension gradient, and $\beta$ is constant contact angle between the fluid and the substrate at the unknown position of the three-phase contact line.

Since the flow is unidirectional, kinematic condition is identically satisfied.

With the respect to the angle $\varphi$ to the horizontal we consider the case $0 < \varphi < \pi/2$. In this case with a triangular section we have $H(y) = A\,|y|$.

Then the solution is as follows:

- the velocity

$$u(y,z) = \tfrac{\rho g \sin\varphi}{2\mu}\big(z - A|y|\big)\big(2h + A\,|y| - z\big) + \tfrac{1}{\mu}\Gamma_x\big(z - A|y|\big),$$

- the pressure

$$p(z) = p_0 - \rho g z\cos\varphi + \beta\big(\bar{\Gamma}\rho g|\cos\varphi|\big)^{1/2}\coth\Big(\big(\tfrac{\rho g|\cos\varphi|}{\bar{\Gamma}}\big)^{1/2} a\Big), \quad 0 < \varphi < \pi/2,$$

- the free-surface profile $z = (H+h)(y)$, with $H(y) = A\,|y|$, is given by

$$(H+h)(y) = \beta\left(\tfrac{\bar{\Gamma}}{\rho g|\cos\varphi|}\right)^{1/2}\frac{\mathrm{ch}\Big(\frac{\rho g|\cos\varphi|}{\bar{\Gamma}}\Big)^{1/2} a - \mathrm{ch}\Big(\frac{\rho g|\cos\varphi|}{\bar{\Gamma}}\Big)^{1/2} y}{m\,\mathrm{sh}\Big(\frac{\rho g|\cos\varphi|}{\bar{\Gamma}}\Big)^{1/2} a}, \quad 0 < \varphi < \pi/2,$$

where we have introduced the notation $m = \sqrt{|\cos\varphi|}$ .

The volume flux of liquid running the substrate is

$$Q = \tfrac{\rho g\sin\varphi}{3}\int_{-a}^{a} h^3(y)\mathrm{d}y + \tfrac{1}{2\mu}\Gamma_x\int_{-a}^{a} h^2(y)\mathrm{d}y.$$

At this point it is convenient to introduce appropriate nondimensional variables defined by:

$$x = \tfrac{1}{\beta}\left(\tfrac{\bar{\Gamma}}{\rho g}\right)^{1/2} x^*,\ y = \left(\tfrac{\bar{\Gamma}}{\rho g}\right)^{1/2} y^*,\ z = \beta\left(\tfrac{\bar{\Gamma}}{\rho g}\right)^{1/2} z^*,\ u = \tfrac{\beta^2\bar{\Gamma}}{\mu}u^*,$$

$$Q = \tfrac{\beta^3\bar{\Gamma}^2}{\rho g\mu}Q^*,\ h = \beta\left(\tfrac{\bar{\Gamma}}{\rho g}\right)^{1/2} h^*, H = \beta\left(\tfrac{\bar{\Gamma}}{\rho g}\right)^{1/2} H^*,\ a = \left(\tfrac{\bar{\Gamma}}{\rho g}\right)^{1/2} a^*,\ \Gamma = \bar{\Gamma}\ \Gamma^*$$

and so, dropping the star superscripts, we yield the following solution for $Q$:

$$
\begin{aligned}
Q =& \\
=& \frac{\sin\varphi}{3}\frac{1}{6m^4}\Big[A^3\Big(12ma\,\mathrm{cth}^3\,ma - 24\,\mathrm{cth}^2\,ma - 18ma\,\mathrm{cth}^2\,ma - 18m^2a^2\,\mathrm{cth}^2\,ma \\
&\qquad + 18m\,\mathrm{cth}^2\,ma - 12m^3a^3\,\mathrm{cth}\,ma + 72ma\,\mathrm{cth}\,ma \\
&\qquad + 18m^2a\,\mathrm{cth}\,ma + 18ma + 18m + 3m^4a^4 - 44\Big) \\
&\quad + 6A^2\Big(6ma\,\mathrm{cth}^3\,ma - 15\,\mathrm{cth}^2\,ma + 9ma\,\mathrm{cth}^2\,ma \\
&\qquad - 6m^2a^2\,\mathrm{cth}^2\,ma + 18\,\mathrm{cth}^2\,ma - 3m\,\mathrm{cth}^2\,ma \\
&\qquad + 2m^3a^3\,\mathrm{cth}\,ma - 12ma\,\mathrm{cth}\,ma - 3m^2a\,\mathrm{cth}\,ma - 9ma \\
&\qquad - 3m + 6m^2a^2 - 2\Big) \\
&\quad + 2A\Big( - 18ma\,\mathrm{cth}^3\,ma + 18\,\mathrm{cth}^2\,ma - 27ma\,\mathrm{cth}^2\,ma \\
&\qquad + 9m^2a^2\,\mathrm{cth}^2\,ma + 27ma + 9m^2a^2 - 3\Big) \\
&\quad + 2\Big(6ma\,\mathrm{cth}^3\,ma - 15\,\mathrm{cth}^2\,ma + 9m^2a^2\,\mathrm{cth}^2\,ma - 9ma + 4\Big)\Big] \\
&+ \frac{1}{2}\Gamma_x\frac{1}{3m^3}\Big[A^2\Big(9ma\,\mathrm{cth}^2\,ma + 6m^2a^2\,\mathrm{cth}\,ma + 6\,\mathrm{cth}\,ma + 2m^3a^3 - 3ma\Big) \\
&\quad + 3A\Big(2m^2a^2\,\mathrm{cth}^2\,ma + 2\,\mathrm{cth}\,ma - 6ma\,\mathrm{cth}^2\,ma + 2ma\Big) \\
&\quad + 3\Big(3ma\,\mathrm{cth}^2\,ma - 3\,\mathrm{cth}\,ma - ma\Big)\Big].
\end{aligned}
$$

## 3. NUMERICAL RESULTS

If $H(y) = A\,|y|$ and $0 \le \varphi < \pi/2$ then $h(y) = (1 - A\,\mathrm{sgn}\,y)f(y) + A(a - |y|)$, where the function $f = f(y)$ is defined by $f(y) = \frac{\cosh ma - \cosh my}{m\sinh ma}$, if $0 \le \varphi < \frac{\pi}{2}$.

We calculated the values of the volume flux for the next numerical values: $2a = 0.0387(\mathrm{cm})$, $m = 0.8752396489$, $A = 0.0517$, $\varphi = 40°$.

If $\Gamma_x = 30.888689(\mathrm{mN/m})$ then we obtain for the volume flux the numerical value $Q^* = 0.0006$.

If $\Gamma_x = 30.88869(\mathrm{mN/m})$ then $Q^* = 0.0008$.

In this case we compare the value of the volume flux given by [4], namely $Q^* = 0.0007$, with our values. We observe that the error is $\mathcal{O}(10^{-4})$.

## 4. ASYMPTOTIC ANALYSIS FOR $H \neq 0$

**A)** The limit $a \to 0$.

In the limit of a narrow rivulet, $a \to 0$, gravity effects become insignificant and the behavior of $Q$ depends only on the local behavior of $H$ near $y = 0$.

In particular, if $H(y) = \mathcal{O}(y)$, as $a \to 0$, then $h(y) \sim \frac{a^2-y^2}{2a}$, as $a \to 0$, and hence $Q \sim \frac{4a^4}{105} \sin\varphi + \Gamma_x \frac{a^3}{15}$, in agreement with the corresponding result in the special case $H \sim 0$.

Alternatively, if $H(y) \sim A\,|y|$, as $y \to 0$, then

$$h(y) \sim (1 - A\,\mathrm{sgn}(y))\tfrac{a^2-y^2}{2a} + A(a - |y|), \quad \text{as } a \to 0,$$

and the hence

$$Q \sim \tfrac{a^4}{420} \sin\varphi \left(147A^3 + 174A^2 + 77A + 16\right) + \tfrac{1}{2}\Gamma_x \tfrac{a^3}{30} \left(28A^2 + 25A + 8\right).$$

**B)** The limit $\varphi \to 0^+$.

In the limit $\varphi \to 0^+$ solutions satisfying $Q = \bar{Q}$ are possible only if $a \to \infty$ (i.e., if the rivulet becomes infinitely wide).

When $0 \le \varphi \le \pi/2$ the behavior of $Q$ as $a \to \infty$ depends on the behavior of $H(y)$, as $|y| \to \infty$. If $H(y) = A\,|y|$, with $A \neq 0$, then $h(y) \sim A(a - |y|)$ and hence

$$Q \sim \tfrac{a^4}{6} \sin\varphi A^3 + \Gamma_x \tfrac{a^3}{3} A^2, \quad \text{as } a \to \infty.$$

## 5. DISCUSSION

If $H(y) = A\,|y|$ then the profile is always convex when $A < 1$, and always concave when $A > 1$.

The case $A = 1$ is exceptional. In this case $(H + h)(y) = H(a) = a$, i.e, the transverse rivulet profile is identically flat for all values of $\varphi$. As a consequence, the exact expression for $Q$ is particulary simple:

$$Q = \tfrac{207a^4}{210} \sin\varphi + \Gamma_x \tfrac{61a^3}{60}.$$

If $\Gamma_x = 0$ then we can write an explicit expression for $a$, namely

$$a = \left(\tfrac{210Q}{207 \sin\varphi}\right)^{1/4}.$$

## 6. CONCLUSIONS

In the lubrication approximation we have considered the flow in an inclined channel of triangular section. Consequently, the flow is driven simultaneously by gravity and a surface tension gradient. This gradient implies a non-zero tangential stress boundary condition (Marangoni effect). We have estimated the respond of the fluid to such stress. The extra terms that correspond to this boundary condition are in perfect agreement with the physics of the flow. More than that, in some particular situations we have estimated them numerically. There is a fairly good agreement between our computed values of the volume flux of the fluid and those reported in [4].

## REFERENCES


[1] Acheson, D. J., *Elementary Fluid Dynamics*, Oxford University Press, Oxford, United Kingdom, pp. 238–259, 1990.

[2] Allen, R. F. and Biggin, C. M., *Stability regions of nonlinear autonomous systems*, IEEE Trans. Aut. Control, **33**, pp. 16–27, 1988.

[3] Braun, R. F. and Murray, B. T., *Lubrication theory for reactive spreading of a thin drop*, Phys. Fliuds, **7**, 1995.

[4] Chifu, E., Gheorghiu, C. I. and Stan, I., *Surface mobility of surfactant solution. Numerical analysis for the Marangoni and gravity flow in a thin liquid layer of triangular section*, Rev. Roumaine Chim., **29**, pp. 31–42, 1984.

[5] Duffy, B. R. and Moffatt, H. K., *Flow of a viscous trickle on a slowly varying*, The Chemical Engeneering Journal, **60**, pp. 141–146, 1995.

[6] Duffy, B. R. and Wilson, S. K., *A third-order differential equation arising in thin-film flows and relevant to Tanner's law*, Appl. Math. Lett., **10**, no. 3, pp. 63–68, 1997.

[7] Ockendon, H. and Ockendon, J. R., *Viscous Flow*, Cambridge University Press, 1995.

[8] Towel, G. D. and Rothfeld, L. B., *Hydrodynamics of rivulet flow*, AICHE. J., 12, 1972.

[9] Wilson, S. K. and Terrrill, E. L., *The dynamics of planar and axisymmetric holes in thin fluid layers*, in First European Coating Symposium on The Mechanics of Thin Film Coatings, Leeds University, U.K., edited by Gaskell, P. H., Savage, 1995.

[10] Young, G. W. and Davis, S. H., *Rivulet instabilities*, J. Fluid Mech., **176**, no. 1, 1987.


Received by the editors: May 25, 2000.