REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 33 (2004) no. 1, pp. 3–9
ictp.acad.ro/jnaat

RATE OF CONVERGENCE OF STANCU BETA OPERATORS FOR FUNCTIONS OF BOUNDED VARIATION

ULRICH ABEL* and VIJAY GUPTA†

Abstract. In this paper we study beta operators of second kind recently introduced by Prof. D. D. Stancu. We obtain an estimate on the rate of convergence for functions of bounded variation by means of the decomposition technique.

MSC 2000. 41A36, 41A25, 41A30, 26A45.

Keywords. Approximation by positive operators, rate of convergence, degree of approximation, functions of bounded variation, total variation.

1. INTRODUCTION

In order to approximate Lebesgue integrable functions on the interval $I = (0, \infty)$, in 1995 D. D. Stancu [4] defined beta operators L_n of second kind given by

$$(L_n f)(x) = \tfrac{1}{B(nx,n+1)} \int_0^\infty \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t)\,\mathrm{d}t \tag{1}$$

and investigated their approximation properties. Obviously the operators L_n are positive linear operators on the space of locally integrable functions on I of polynomial growth as $t \to \infty$, provided that n is sufficiently large. The fact that the operators (1) preserve linear functions is advantageous for their approximation properties. Recently Abel [1] derived the complete asymptotic expansion for the sequence of these operators.

In the present paper we study the rate of convergence of the operators L_n by deriving an estimate of $\left| (L_n f)(x) - \frac{1}{2}\{f(x+) + f(x-)\} \right|$ for functions f of bounded variation (see Theorem 1).

For the sake of a convenient notation in the proofs we rewrite the operators (1) as

$$(L_n f)(x) = \int_0^\infty K_n(x,t)\ f(t)\ \mathrm{d}t, \tag{2}$$

where the kernel function K_n is given by

$$K_n(x,t) = \tfrac{1}{B(nx,n+1)} \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}}. \tag{3}$$

*Fachhochschule Giessen-Friedberg, University of Applied Sciences, Fachbereich MND, Wilhelm-Leuschner-Straße 13, 61169 Friedberg, Germany, e-mail: Ulrich.Abel@mnd.fh-friedberg.de.

†School of Applied Sciences, Netaji Subhas Institute of Technology, Azad Hind Fauj Marg, Sector-3, Dwarka, New Delhi–110045, India, e-mail: vijay@nsit.ac.in.

Throughout this note, for fixed $x \in I$, we use the auxiliary function f_x, which is defined by

$$f_x(t) = \begin{cases} f(t) - f(x-), & 0 \le t < x, \\ 0, & t = x, \\ f(t) - f(x+), & x < t < \infty. \end{cases} \tag{4}$$

The following theorem is our main result.

Theorem 1. *Let $r > 0$ and let f be a function of bounded variation on each finite subinterval of I satisfying $f(t) = \mathcal{O}(t^r)$, as $t \to \infty$. Fix a point $x \in I$. Then for each $\varepsilon > 0$ there exists an integer $n(\varepsilon)$, such that for all $n \ge n(\varepsilon)$ there holds*

$$\left| L_n(f,x) - \tfrac{1}{2}\{f(x+) + f(x-)\} \right| \le$$

$$\le |f(x+) - f(x-)| \tfrac{1+\varepsilon+2x}{3\sqrt{2\pi n x(1+x)}} + \tfrac{2+3x}{(n-1)x} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (f_x) + \tfrac{M(f,x,r)}{n^r},$$

where $M(f,x,r)$ is a constant independent of n and $\bigvee_a^b (f_x)$ denotes the total variation of f_x on $[a,b]$.

The proof of Theorem 1 is contained in Section 3, while the next section contains some auxiliary results used in the proof.

2. AUXILIARY RESULTS

In this section we give several results, which are necessary to prove Theorem 1.

For fixed $x \in \mathbb{R}$, define the function ψ_x, by $\psi_x(t) = t - x$. The first central moments for the operators L_n are given by

$$\left(L_n \psi_x^0\right)(x) = 1, \quad \left(L_n \psi_x^1\right)(x) = 0, \quad \left(L_n \psi_x^2\right)(x) = \tfrac{x(1+x)}{n-1} \tag{5}$$

(see [1, Proposition 2]). In general we have the following result:

Lemma 2. [1, Proposition 2] *Let fixed $x \in I$ be fixed. For $s = 0, 1, 2, \ldots$ and $n \in \mathbb{N}$, the central moments for the operators L_n satisfy*

$$\left(L_n \psi_x^s\right)(x) = \mathcal{O}\big(n^{-\lfloor (s+1)/2 \rfloor}\big), \qquad \text{as } n \to \infty.$$

Lemma 3. *Let $x \in I$ and $K_n(x,t)$ be defined by Eq. (3). Then for $n \ge 2$, we have*

$$\lambda_n(x,y) = \int_0^y K_n(x,t)\,\mathrm{d}t \le \tfrac{x(1+x)}{(n-1)(x-y)^2}, \qquad 0 \le y < x, \tag{i}$$

and

$$1 - \lambda_n(x,z) = \int_z^\infty K_n(x,t)\,\mathrm{d}t \le \tfrac{x(1+x)}{(n-1)(z-x)^2}, \qquad x < z < \infty. \tag{ii}$$

Proof. First, we prove (i). In view of Eq. (5), we have

$$\int_0^y K_n(x,t)\,\mathrm{d}t \le \int_0^y \tfrac{(x-t)^2}{(x-y)^2} K_n(x,t)\,\mathrm{d}t \le (x-y)^{-2} L_n\left(\psi_x^2, x\right) \le \tfrac{x(1+x)}{(n-1)(x-y)^2}.$$

The proof of (ii) is similar. □

LEMMA 4. *For each $x>0$, we have*

$$\tfrac{1}{B(nx,n+1)} \int_0^x \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}}\mathrm{d}t =$$

$$= \tfrac{1}{2} + \tfrac{1+2x}{3\sqrt{2\pi nx(1+x)}} - \tfrac{(1+2x)(-1+23x+23x^2)}{540\sqrt{2\pi}(nx(1+x))^{3/2}} + \mathcal{O}\left(n^{-2}\right), \quad \text{as } n\to\infty.$$

Proof. Let $x>0$, and put

$$I_x(n) = \int_0^x \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}}\mathrm{d}t.$$

By a change of variable we obtain

$$I_x(n) = \int_{1/x}^{\infty} \tfrac{t^n}{(1+t)^{nx+n+1}}\mathrm{d}t = \int_{1/x}^{\infty} (1+t)^{-1} \mathrm{e}^{nh_x(t)}\mathrm{d}t,$$

where

$$h_x(t) = \log t - (1+x)\log(1+t).$$

Since

$$h_x'(t) = \tfrac{1-tx}{t(1+t)} < 0, \quad t > 1/x, \quad \text{and} \quad h_x'(1/x) = 0,$$

h_x is strictly decreasing on $(1/x,\infty)$. Furthermore,

$$h_x''(t) = \tfrac{x\left(t-x^{-1}\right)^2 - x^{-1}(1+x)}{t^2(1+t)^2}, \qquad \text{and} \qquad h_x''(1/x) = -x^3(1+x)^{-1} \ne 0.$$

Thus it is well-known (e.g., $I_x(n)$ meets the assumptions of [2, Theorem 1, Kap. 3, §5]), that there holds the complete asymptotic expansion

$$\begin{aligned} I_x(n) \quad &\sim \quad \tfrac{1}{2} e^{nh_x(1/x)} \sum_{k=0}^{\infty} a_k \tfrac{\Gamma((k+1)/2)}{n^{(k+1)/2}} \\ &= \quad \left(\tfrac{x^x}{(1+x)^{1+x}}\right)^n \left[\tfrac{a_0\sqrt{\pi}}{2n^{1/2}} + \tfrac{a_1}{2n} + \tfrac{a_2\sqrt{\pi}}{4n^{3/2}} + \tfrac{a_3}{2n^2} + \tfrac{3a_4\sqrt{\pi}}{8n^{5/2}} + \mathcal{O}\left(n^{-3}\right)\right] \end{aligned}$$

for $n\to\infty$ with the coefficients

$$a_k = \tfrac{1}{k!}\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k \left[(1+t)^{-1}\left(\tfrac{t-1/x}{\sqrt{h_x(1/x)-h_x(t)}}\right)^{k+1}\right]\right)\Big|_{t=1/x}.$$

By direct calculation we obtain the explicit expressions

$$\begin{aligned} a_0 \quad &= \quad \sqrt{\tfrac{2}{x(1+x)}}, \qquad a_1 = \tfrac{2(1+2x)}{3x(1+x)}, \qquad a_2 = \tfrac{1+x+x^2}{3\sqrt{2}(x(1+x))^{3/2}}, \\ a_3 \quad &= \quad \tfrac{4(1-x)(x+2)(1+2x)}{135(x(1+x))^2}, \qquad a_4 = \tfrac{\left(1+x+x^2\right)^2}{108\sqrt{2}(x(1+x))^{5/2}}. \end{aligned}$$

Thus we have

$$I_x(n) = \left(\frac{x^x}{(1+x)^{1+x}}\right)^n \left[\sqrt{\frac{\pi}{2nx(1+x)}} + \frac{1+2x}{3nx(1+x)} + \frac{\sqrt{\pi}}{12\sqrt{2}}\frac{1+x+x^2}{n^{3/2}(x(1+x))^{3/2}} \right.$$
$$\left. + \frac{2}{135}\frac{(1-x)(x+2)(1+2x)}{n^2(x(1+x))^2} + \mathcal{O}\left(n^{-5/2}\right)\right].$$

On the other hand, application of Stirling's formula yields

$$\begin{aligned}\frac{1}{B(nx,n+1)} &= (1+x)\frac{\Gamma(n(1+x))}{\Gamma(nx)\Gamma(n)} \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{(1+x)^{1+x}}{x^x}\right)^n \\ &\quad \times \left[\sqrt{nx(1+x)} - \frac{1+x+x^2}{12\sqrt{nx(1+x)}} + \frac{\left(1+x+x^2\right)^2}{288n^{3/2}(x(1+x))^{3/2}} + \mathcal{O}\left(n^{-5/2}\right)\right].\end{aligned}$$

Combination of both asymptotic expressions proves Lemma 4. □

3. PROOF OF THE MAIN THEOREM

We close the paper by giving the proof of the main theorem.

Proof of Theorem 1. Let $x \in I$. Our starting-point is the inequality

$$\left|L_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\}\right| \leq \tag{6}$$
$$\leq |(L_n f_x)(x)| + \frac{1}{2}|f(x+) - f(x-)| \cdot |(L_n \operatorname{sign}_x)(x)|,$$

where sign_x is defined by $\operatorname{sign}_x(t) = \operatorname{sign}(t-x)$.

In order to prove the theorem we need estimates for $(L_n f_x)(x)$ and $(L_n \operatorname{sign}_x)(x)$. We first estimate $(L_n \operatorname{sign}_x)(x)$ as follows:

With the notation (2) we obtain

$$(L_n \operatorname{sign}_x)(x) = \int_x^\infty K_n(x,t)\,\mathrm{d}t - \int_0^x K_n(x,t)\,\mathrm{d}t = 1 - 2\int_0^x K_n(x,t)\,\mathrm{d}t$$

since

$$\int_0^\infty K_n(x,t)\,\mathrm{d}t = 1.$$

By Lemma 4, for each $\varepsilon > 0$ there exists an integer $n = n(\varepsilon)$, such that

$$|(L_n \operatorname{sign}_x)(x)| \leq \frac{2(1+\varepsilon+2x)}{3\sqrt{2\pi nx(1+x)}}, \qquad n \geq n(\varepsilon). \tag{7}$$

Next we estimate $L_n(f_x,x)$ as follows:

$$\begin{aligned}(L_n f_x)(x) &= \int_0^\infty K_n(x,t) f_x(t)\,\mathrm{d}t \\ &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) K_n(x,t) f_x(t)\,\mathrm{d}t =: E_1 + E_2 + E_3,\end{aligned} \tag{8}$$

say, where $I_1 = [0, x - x/\sqrt{n}]$, $I_2 = [x - x/\sqrt{n}, x + x/\sqrt{n}]$ and $I_3 = [x + x/\sqrt{n}, \infty)$.

We first estimate E_1. Writing $y = x - x/\sqrt{n}$ and

$$\lambda_n(x,y) = \int_0^y K_n(x,t)\,\mathrm{d}t$$

we have, integrating by parts,

$$E_1 = \int_0^y f_x(t)\,\mathrm{d}_t(\lambda_n(x,t)) = f_x(y)\,\lambda_n(x,y) - \int_0^y \lambda_n(x,t)\,\mathrm{d}_t(f_x(t)).$$

Since $|f_x(y)| \le \bigvee_y^x (f_x)$, it follows that

$$|E_1| \le \bigvee_y^x (f_x)\,\lambda_n(x,y) + \int_0^y \lambda_n(x,t)\,\mathrm{d}_t\Big(-\bigvee_t^x (f_x)\Big).$$

By using (i) of Lemma 3, we get

$$|E_1| \le \bigvee_y^x (f_x)\,\tfrac{x(1+x)}{(n-1)(x-y)^2} + \tfrac{x(1+x)}{n-1}\int_0^y \tfrac{1}{(x-t)^2}\mathrm{d}_t\Big(-\bigvee_t^x (f_x)\Big).$$

Integrating by parts the last term we have

$$|E_1| \le \tfrac{x(1+x)}{n-1}\left[x^{-2}\bigvee_0^x (f_x) + 2\int_0^y \tfrac{\bigvee_t^x (f_x)}{(x-t)^3}\mathrm{d}t\right].$$

Now replacing the variable y in the last integral by $x - x/\sqrt{n}$, we obtain

$$\int_0^{x-x/\sqrt{n}} \tfrac{\bigvee_t^x (f_x)}{(x-t)^3}\mathrm{d}t = \sum_{k=1}^{n-1}\int_{x/\sqrt{k}}^{x/\sqrt{k+1}} \bigvee_{x-t}^x (f_x)\, t^{-3}\mathrm{d}t \le \tfrac{1}{2x^2}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^x (f_x)$$

and therefore,

$$|E_1| \le \tfrac{1+x}{(n-1)x}\Big[\bigvee_0^x (f_x) + \sum_{k=1}^{n-1}\bigvee_{x-x/\sqrt{k}}^x (f_x)\Big] \le \tfrac{2(1+x)}{(n-1)x}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^x (f_x). \tag{9}$$

Next we estimate E_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|f_x(t)| = |f_x(t) - f_x(x)| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (f_x) \le \tfrac{1}{n}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (f_x)$$

and, since $\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} \mathrm{d}_t(\lambda_n(x,t)) \le 1$, we conclude that

$$|E_2| \le \tfrac{1}{n}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (f_x). \tag{10}$$

Finally, we estimate E_3. We put

$$\widetilde{f}_x(t) = \begin{cases} f_x(t), & 0 \le t \le 2x, \\ f_x(2x), & 2x < t < \infty, \end{cases}$$

and divide $E_3 = E_{31} + E_{32}$, where

$$E_{31} = \int_{x+x/\sqrt{n}}^{\infty} K_n(x,t)\ \widetilde{f}_x(t)\ \mathrm{d}t,$$

$$E_{32} = \int_{2x}^{\infty} K_n(x,t)\ [f_x(t) - f_x(2x)]\ \mathrm{d}t.$$

With $y = x + x/\sqrt{n}$ the first integral can be written in the form

$$E_{31} = \lim_{R\to+\infty} \Big\{ f_x(y)[1-\lambda_n(x,y)] + \widetilde{f}_x(R)[\lambda_n(x,R)-1] + \int_y^R [1-\lambda_n(x,t)]\ \mathrm{d}_t \widetilde{f}_x(t) \Big\}.$$

By Eq. (ii) of Lemma 3, we conclude, for sufficiently large n,

$$\begin{aligned} |E_{31}| &\le \tfrac{x(1+x)}{n-1} \lim_{R\to+\infty} \left\{ \tfrac{\bigvee_x^y(f_x)}{(y-x)^2} + \tfrac{\left|\widetilde{f}_x(R)\right|}{(R-x)^2} + \int_y^R \tfrac{1}{(t-x)^2}\ \mathrm{d}_t\Big(\bigvee_x^t(\widetilde{f}_x)\Big) \right\} \\ &= \tfrac{x(1+x)}{n-1} \left\{ \tfrac{\bigvee_x^y(f_x)}{(y-x)^2} + \int_y^{2x} \tfrac{1}{(t-x)^2}\ \mathrm{d}_t\Big(\bigvee_x^t(f_x)\Big) \right\}. \end{aligned}$$

Using the similar method as above we obtain

$$\int_y^{2x} \tfrac{1}{(t-x)^2}\ \mathrm{d}_t\Big(\bigvee_x^t(f_x)\Big) \le x^{-2} \bigvee_x^{2x}(f_x) - \tfrac{\bigvee_x^y(f_x)}{(y-x)^2} + x^{-2} \sum_{k=1}^{n-1} \bigvee_x^{x+x/\sqrt{k}} (f_x)$$

which implies the estimate

$$|E_{31}| \le \tfrac{2(1+x)}{(n-1)x} \sum_{k=1}^{n} \bigvee_x^{x+x/\sqrt{k}} (f_x). \tag{11}$$

Lastly, we estimate E_{32}. By assumption, there exists an integer r, such that $f(t) = \mathcal{O}(t^{2r})$, as $t\to\infty$. Thus, for a certain constant $C>0$ depending only on f, x and r, we have

$$|E_{32}| \le C \int_{2x}^{\infty} K_n(x,t)\, t^{2r}\ \mathrm{d}t.$$

Obviously, $t \ge 2x$ implies $t \le 2(t-x)$ and it follows

$$|E_{32}| \le 2^{2r}\left(L_n \psi_x^{2r}\right)(x).$$

By Lemma 2, we arrive at

$$E_{32} = \mathcal{O}(n^{-r}), \qquad \text{as } n\to\infty. \tag{12}$$

Collecting the estimates (9)–(12), we have, by Eq. (8),

$$|(L_n f_x)(x)| \le \left(\tfrac{2(1+x)}{(n-1)x} + \tfrac{1}{n}\right) \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (f_x) + \mathcal{O}(n^{-r}). \tag{13}$$

Combining the estimates of (6), (7) and (13) completes the proof of the theorem. □

REFERENCES

[1] Abel, U., *Asymptotic approximation with Stancu Beta operators*, Rev. Anal. Numér. Théor. Approx., **27**, no. 1, pp. 5–13, 1998.

[2] Doetsch, G., *Handbuch der Laplace-Transformation, vol. II*, Birkhäuser, Basel and Stuttgart, 1972.

[3] Khan, M. K., *Approximation at discontinuity*, Suppl. Rend. Circ. Mat. Palermo, II. Ser., **68**, no. 1, pp. 539–553, 2002.

[4] Stancu, D. D., *On the beta approximating operators of second kind*, Rev. Anal. Numér. Théor. Approx., **2**, no. 4, pp. 231–239, 1995.

Received by the editors: October 16, 2003.