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LOCAL CONVERGENCE OF GENERAL STEFFENSEN TYPE METHODS*

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Abstract. We study the local convergence of a generalized Steffensen method. We show that this method substantially improves the convergence order of the classical Steffensen method. The convergence order of our method is greater or equal to the number of the controlled nodes used in the Lagrange-type inverse interpolation, which, in its turn, is substantial higher than the convergence orders of the Lagrange type inverse interpolation with uncontrolled nodes (since their convergence order is at most 2).

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1. INTRODUCTION

In this paper we study the local convergence of some general methods of Aitken-Steffensen type, which are based on inverse interpolation of Lagrange type.

Let $f:[a,b]\to\mathbb{R}$, $a,b\in\mathbb{R}$, $a<b$ be a function and x_i, $i=\overline{0,n}$, $n+1$ distinct points in $[a,b]$, which we call interpolation nodes. Denote $y_i=f(x_i)$, $i=\overline{0,n}$, and suppose that $y_i\neq y_j$ for $i\neq j$. Assume in the beginning that $f:I\to f(I)$, $I=[a,b]$ is one-to-one, i.e., there exists $f^{-1}:f(I)\to I$. Consider the Lagrange polynomial with the interpolation nodes y_i, $i=\overline{0,n}$ and the values of f^{-1} on these nodes $x_i=f^{-1}(y_i)$, $i=\overline{0,n}$. This is the inverse interpolation polynomial, which we denote by $L(y_0,y_1,...,y_n;f^{-1}|\,y)$, and it can be represented in the Lagrange form

$$(1)\qquad L\left(y_0,y_1,...,y_n;f^{-1}|\,y\right)=\sum_{i=0}^{n}\tfrac{x_i\omega(y)}{(y-y_i)\omega'(y_i)},\quad \omega(y)=\prod_{i=0}^{n}(y-y_i)$$

and in the Newton form:

$$(2)\qquad \begin{aligned} L\left(y_0,y_1,...,y_n;f^{-1}|\,y\right)=&x_0+\left[y_0,y_1;f^{-1}\right](y-y_0)\\ &+\left[y_0,y_1,y_2;f^{-1}\right](y-y_0)(y-y_1)+\cdots\\ &+\left[y_0,y_1,...,y_n;f^{-1}\right](y-y_0)(y-y_1)\cdots(y-y_{n-1}),\end{aligned}$$

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where $\left[y_0,...,y_i;f^{-1}\right], i=\overline{1,n}$ denotes the i-th order divided difference of the function f^{-1} on the nodes $y_0,...,y_i$.

Assuming that f admits derivatives up to the order $n+1$ on the interval $[a,b]$, then

$$f^{-1}(y)=L\left(y_0,y_1,...,y_n;f^{-1}|y\right)+\tfrac{\left[f^{-1}(\xi)\right]^{(n+1)}}{(n+1)!}\omega(y) \tag{3}$$

where ξ is a point belonging to the smallest interval containing $y,y_0,...,y_n$. Denote

$$R_n\left[f^{-1};y\right]=\tfrac{\left[f^{-1}(\xi)\right]^{(n+1)}}{(n+1)!}\omega(y).$$

Consider now the equation

$$f(x)=0. \tag{4}$$

If it has a solution $\overline{x}\in[a,b]$, then obviously

$$\overline{x}=f^{-1}(0). \tag{5}$$

An approximation of the solution $\overline{x}$ can be obtained from (3) for $y=0$, i.e.,

$$\overline{x}=L\left(y_0,y_1,...,y_n;f^{-1}|0\right)+R_n\left[f^{-1};0\right], \tag{6}$$

whence, by neglecting the remainder $R_n\left[f^{-1},0\right]$ we get

$$x_{n+1}=L\left(y_0,y_1,...,y_n;f^{-1}|0\right), \tag{7}$$

and the error

$$\overline{x}-x_{n+1}=R_n\left[f^{-1};0\right]. \tag{8}$$

Denoting $M=\sup\limits_{y\in f(I)}\left|\left[f^{-1}(y)\right]^{(n+1)}\right|$, then

$$\left|\overline{x}-x_{n+1}\right|\le\tfrac{M}{(n+1)!}\left|y_0\right|\left|y_1\right|\cdots\left|y_n\right|. \tag{9}$$

Assuming that $x_{n+1}\in[a,b]$ and denoting $y_{n+1}=f(x_{n+1})$, then we can obtain a new approximation x_{n+2} given by relation

$$x_{n+2}=L\left(y_1,y_2,...,y_n,y_{n+1};f^{-1}|0\right), \tag{10}$$

where, as it can be seen, the node x_0 has been neglected and instead we consider y_{n+1}. The above procedure may continue indefinitely: assuming that we have obtained the approximations $x_k,x_{k+1},...,x_{k+n}\in[a,b]$ then the next approximation is given by

$$x_{n+k+1}=L\left(y_k,y_{k+1},...,y_{n+k};f^{-1}|0\right),\ k=0,1,..., \tag{11}$$

where $y_{k+i}=f(x_{k+i})$, $i=\overline{0,n}$. If all the iterates are contained in $[a,b]$, then the procedure may continue indefinitely.

In the same way as for (9), we get the following error bound:

$$\left|\overline{x}-x_{n+k+1}\right|\le\tfrac{M}{(n+1)!}\left|y_k\right|\left|y_{k+1}\right|\cdots\left|y_{k+n}\right|,\quad k=0,1,..., \tag{12}$$

Assume that $f'(x) \neq 0, \forall x \in [a,b]$, and denote

$$m = \sup_{x \in I} \left| f'(x) \right|.$$

Obviously

$$|\overline{x} - x| \geq \frac{|f(x)|}{m}. \tag{13}$$

By (12) and (13)

$$|f(x_{n+k+1})| \leq \tfrac{mM}{(n+1)!} |f(x_k)| \, |f(x_{k+1})| \cdots |f(x_{k+n})|, \quad k = 0, 1, \ldots . \tag{14}$$

Multiplying relations (14) by $\left(\frac{mM}{(n+1)!}\right)^{\frac{1}{n}}$ and denoting

$$\rho_i = \left(\tfrac{mM}{(n+1)!}\right)^{\frac{1}{n}} |f(x_i)|, \quad i = 0, 1, \ldots$$

leads to

$$\rho_{n+k+1} \leq \rho_k \rho_{k+1} \ldots \rho_{k+n}, \quad k = 0, 1, \ldots. \tag{15}$$

Suppose now that $\rho_i \leq d^{\alpha_i}$, with $0 < d < 1$ and $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$, $i = \overline{0, n}$. Then

$$\rho_{n+1} \leq d^{\alpha_0 + \alpha_1 + \cdots + \alpha_n} = d^{\alpha_{n+1}}, \tag{16}$$

where

$$\alpha_{n+1} = \alpha_n + \alpha_{n-1} + \cdots + \alpha_1 + \alpha_0. \tag{17}$$

In general, from (15) it follows

$$\rho_{n+k+1} \leq d^{\alpha_{n+k+1}}, \tag{18}$$

where

$$\alpha_{n+k+1} = \alpha_{n+k} + \alpha_{n+k-1} + \cdots + \alpha_{k+1} + \alpha_k, \quad n = 0, 1, \ldots. \tag{19}$$

Let now $t_0 > 0$ be the unique positive solution of equation

$$t^{n+1} - t^n - t^{n-1} - \cdots - t - 1 = 0. \tag{20}$$

Assume that the values of f obey

$$\rho_i \leq d^{\alpha t_0^i}, \quad i = \overline{0, n}, \tag{21}$$

for a certain constant $\alpha > 0$, i.e., $\alpha_i = \alpha t_0^i$. Then one can show by induction using (19) that

$$\rho_{n+k+1} \leq d^{\alpha t_0^{n+k+1}}, \quad k = 0, 1, \ldots. \tag{22}$$

In [7] it is shown that t_0 verifies $\frac{2(n+1)}{n+2} < t_0 < 2$. It is clear that the convergence order of the sequence given in (11) is less than 2.

In order to increase the convergence order of the sequence we proceed as follows. Consider the following equation, equivalent to (4):

$$x - g(x) = 0. \tag{23}$$

We shall choose the interpolation nodes in (11) using g, generalizing in this way the Steffensen method.

2. GENERAL METHODS OF STEFFENSEN TYPE

Assume in the beginning that for all $x \in [a,b]$, it follows that $g(x) \in [a,b]$.

Let $u_0 \in [a,b]$ be an initial approximation of the solution $\overline{x}$ of equation (14). We shall use the following notations:

$$x_0 = u_0,\ x_1 = g(x_0),\ x_2 = g(x_1), \ldots,\ x_n = g(x_{n-1}), \tag{24}$$

which, by (7) lead to a new approximation for $\overline{x}$

$$u_1 = L\left(y_0, y_1, ..., y_n; f^{-1}|0\right), \tag{25}$$

where $y_i = f(x_i), i = \overline{0,n}$, x_i being given by (24).

From (9) we get:

$$|\overline{x} - u_1| \leq \tfrac{M}{(n+1)!} |f(x_0)| |f(x_1)| \cdots |f(x_n)|, \tag{26}$$

whence, by (13) we get

$$|\overline{x} - u_1| \leq \tfrac{Mm^{n+1}}{(n+1)!} |\overline{x} - x_0| |\overline{x} - x_1| \cdots |\overline{x} - x_n|. \tag{27}$$

Assume that g obeys the Lipschitz condition on $[a,b]$, i.e. there exists $l > 0$ such that

$$|g(x) - g(y)| \leq l |x - y|, \quad \forall x, y \in [a,b].$$

Under this hypothesis, taking into account (24), we are lead to

$$|\overline{x} - u_1| \leq \tfrac{M \cdot m^{n+1} \cdot l^{\frac{n(n+1)}{2}}}{(n+1)!} |\overline{x} - u_0|^{n+1}. \tag{28}$$

Let now u_1 be the next approximation for $\overline{x}$; then, analogously to (24), we consider in (7) the following values to f^{-1} at the interpolation nodes:

$$x_0 = u_1, \quad x_1 = g(x_0) ..., \quad x_n = g(x_{n-1}). \tag{29}$$

In the same way as above, we obtain the next approximation u_2 for $\overline{x}$, which satisfies

$$|\overline{x} - u_2| \leq \tfrac{M \cdot m^{n+1} \cdot l^{\frac{n(n+1)}{2}}}{(n+1)!} |\overline{x} - u_1|^{n+1}.$$

In general, if u_k is an approximation of $\overline{x}$ and we set

$$x_0 = u_k, \quad x_1 = g(x_0), ..., x_n = g(x_{n-1}),$$

then by (7) we obtain the next approximation u_{k+1}, which satisfies

$$|\overline{x} - u_{k+1}| \leq \tfrac{M \cdot m^{n+1} \cdot l^{\frac{n(n+1)}{2}}}{(n+1)!} |\overline{x} - u_k|^{n+1}, \quad k = 0, 1, \ldots. \tag{30}$$

Denoting $\delta_k = ml^{\frac{n+1}{2}} \left(\frac{Mm}{(n+1)!}\right)^{\frac{1}{n}} |\overline{x} - u_k|$, then from the above relation we deduce

$$\delta_{k+1} \leq \delta_k^{n+1}, \quad k = 0, 1, \ldots,$$

which leads to the conclusion that for $n \geq 1$, method (11) converges superlinearly. Moreover, if x_0 is chosen such that $\delta_0 < 1$ then $\lim_{k\to\infty} \delta_k = 0$ and therefore $\lim_{k\to\infty} u_k = \overline{x}$.

The error at each step is bounded by:

$$|\overline{x} - u_k| \leq m^{-1} l^{-\frac{n+1}{2}} \left(\tfrac{Mm}{(n+1)!}\right)^{-\frac{1}{n}} \delta_0^{(n+1)^k}, \quad k = 1, 2, ... \tag{31}$$

In the following we analyze two particular cases.

(1) Case $n = 1$. In this case (11) leads to the well known Steffensen method.

Indeed, by (2) we get

$$L\left[y_0, y_1 f^{-1} | y\right] = x_0 + \left[y_0, y_1; f^{-1}\right](y - y_0), \tag{32}$$

hence, taking into account the equality

$$\left[y_0, y_1; f^{-1}\right] = \tfrac{1}{[x_0, x_1; f]},$$

for $y = 0$ we obtain the approximation

$$x_2 = x_0 - \tfrac{f(x_0)}{[x_0, x_1; f]}, \tag{33}$$

i.e., the first step in the chord method.

Obviously, (9) may continue by

$$x_k = x_{k-2} - \tfrac{f(x_{k-2})}{[x_{k-2}, x_{k-1}; f]}, \quad k = 2, 3, ... \tag{34}$$

Denoting in (33) $x_0 = u_0$ and $x_1 = g(u_0)$, we get

$$u_1 = u_0 - \tfrac{f(u_0)}{[u_0, g(u_0); f]}$$

and in general

$$u_k = u_{k-1} - \tfrac{f(u_{k-1})}{[u_{k-1}, g(u_{k-1}); f]}, \tag{35}$$

which is precisely the Steffensen method.

In this case, the elements of the sequence $(\delta_k)_{k\geq 0}$ have the form

$$\delta_k = \tfrac{lm^2 M}{2} |\overline{x} - u_k|, \quad k = 0, 1, ..., \quad M = \sup_{y \in f(I)} \left| \left(f^{-1}(y)\right)'' \right|$$

and obey

$$\delta_{k+1} \leq \delta_k^2, \quad k = 0, 1, ...$$

If $\delta_0 < 1$, then obviously

$$\lim_{k\to\infty} \delta_k = 0$$

and hence $\lim x_k = \overline{x}$, with the error

$$|x_k - \overline{x}| \leq \tfrac{2}{lm^2 M} \delta_0^{2^k}, \quad k = 1, 2, \ldots,$$

whence (35) converges quadratically.

2. Case $n = 2$

It can be easily seen that the second order divided difference $[y_0, y_1, y_2; f^{-1}]$ can be expressed as

$$\left[y_0, y_1, y_2; f^{-1}\right] = \tfrac{-[x_0,x_1,x_2;f]}{[x_0,x_1;f][x_0,x;f][x_1,x_2;f]}. \tag{36}$$

By (2) we get

$$L\left[y_0, y_1, y_2; f^{-1}|y\right] = x_0 + \left[y_0, y_1; f^{-1}\right](y - y_0) + \left[y_0, y_1, y_2; f^{-1}\right](y - y_0)(y - y_1).$$

Setting $y = 0$ and taking into account (36) and the corresponding formula for the first order divided difference, we are lead to

$$x_3 = x_0 - \tfrac{f(x_0)}{[x_0,x_1;f]} - \tfrac{[x_0,x_1,x_2;f]f(x_0)f(x_1)}{[x_0,x_1;f][x_1,x_2;f][x_1,x_2;f]}, \tag{37}$$

i.e., to a method correcting the chord method.

In general, a method of type (37) has the form

$$x_{n+3} = x_n - \tfrac{f(x_n)}{[x_n,x_{n+1};f]} - \tfrac{[x_n,x_{n+1},x_{n+2};f]}{[x_n,x_{n+1};f][x_{n+1},x_{n+2};f][x_n,x_{n+2}f]}, \tag{38}$$

$n = 0, 1, \ldots$. If in (37) we control the interpolation nodes as

$$x_0 = u_0,\ x_1 = g(x_0),\ x_2 = g(x_1) = g(g(x_0))$$

we obtain

$$u_1 = u_0 - \tfrac{f(u_0)}{[u_0,g(u_0);f]} - \tfrac{[u_0,g(u_0);g(g(u_0));f]f(u_0)f(g(u_0))}{[u_0,g(u_0);f][u_0,g(g(u_0));f][g(u_0),g(g(u_0));f]}.$$

In general, if u_k is an approximation to $\overline{x}$, then u_{k+1} is given by

$$u_{k+1} = u_k - \tfrac{f(u_k)}{[u_k,g(u_k);f]} - \tfrac{[u_k,g(u_k),g(g(u_k));f]f(u_k)f(g(u_k))}{[u_k,g(u_k);f][u_k,g(g(u_k));f][g(u_k),g(g(u_k));f]}. \tag{39}$$

Denoting $M = \sup_{y\in f(I)} \left|[f^{-1}(y)]'''\right|$, then the error satisfies at each iteration step:

$$|u_k - \overline{x}| \leq \tfrac{\sqrt{6}}{ml^{3/2}\sqrt{mM}} \delta_0^{3^k}, \tag{40}$$

where

$$\delta_0 = \tfrac{ml^{3/2}\sqrt{Mm}}{\sqrt{6}} |\overline{x} - u_0|.$$

Assuming $\delta_0 < 1$, then $\lim_{k\to\infty} u_k = \overline{x}$, with the convergence order at least 3.

Suppose in the following that the function g given by (23) has derivatives up to the p-th order, $p \in \mathbb{N}$, $p \geq 2$, on $[a, b]$ and its derivatives satisfy

$$g^{(i)}(\overline{x}) = 0,\ i = \overline{1, p-1},\ g^{(p)}(\overline{x}) \neq 0. \tag{41}$$

In this case, if the derivative of p-th order in continuous on $[a, b]$ and

$$L = \sup_{x\in[a,b]} \left|g^{(p)}(x)\right|,$$

then for all $x \in [a, b]$ one has

$$|g(\overline{x}) - g(x)| \le \tfrac{L}{p!} |\overline{x} - x|^p . \tag{42}$$

Using the above relation (27) we get

$$|\overline{x} - u_1| \le \tfrac{Mm^{n+1}}{(n+1)!} \left(\tfrac{p!}{L}\right)^{\frac{n+1}{p-1}} \theta_0^{\frac{p^{n+1}-1}{p-1}}, \tag{43}$$

where $\theta_0 = \left(\frac{L}{p!}\right)^{\frac{1}{p-1}} |\overline{x} - u_0| .$

We make the following notations:

$$q = \tfrac{p^{n+1}-1}{p-1};$$

$$K = \tfrac{Mm^{n+1}}{(n+1)!} \left(\tfrac{p!}{L}\right)^{\frac{n+1}{p-1}};$$

$$\varepsilon_0 = K^{\frac{1}{q-1}} \theta_0;$$

$$\varepsilon_1 = K^{\frac{1}{q-1}} |\overline{x} - u_1| .$$

By (43), we are lead to

$$\varepsilon_1 \le \varepsilon_0^q.$$

We obtain the sequence of approximation $(u_s)_{s\ge 0}$ for which, if denoting

$$\varepsilon_s = K^{\frac{1}{q-1}} |\overline{x} - u_s| , \quad s = 1, 2, ..., \tag{44}$$

we get

$$\varepsilon_s \le \varepsilon_0^{q^s}, \quad s = 1, 2, \ldots . \tag{45}$$

Obviously, in this case too, if $\varepsilon_0 < 1$, then by (44) and (45) it follows $\lim\limits_{s\to\infty} u_s = \overline{x}$ and the convergence order is q.

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