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HERMITE BICUBIC SPLINE COLLOCATION METHOD FOR POISSON'S EQUATIONS

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Abstract. In this paper is presented a bicubic spline collocation method for the numerical approximation of the solution of Dirichlet problem for the Poisson's equation. The approximating solution is effectively determined in a bicubic Hermite spline functions space by using a suitable basis constructed as a tensorial product of univariate spline spaces.

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1. INTRODUCTION

Let consider the Dirichlet problem for the Poisson's equation on $\Omega \subset R^2$

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega :=]0,1[\times]0,1[$ and $\partial\Omega$ is the boundary of Ω.

Various methods have been developed for solving the Dirichlet problem (1) numerically. A number of works using finite difference methods shows the efficiency of such methods, but their order of accuracy are very low (see [3], [7], [10]). Higher order accuracy can be achieved using finite element methods (see [3], [7], [11]). There are many finite element approaches which use iterative methods, such as [3], [4].

In this paper we shall present a direct bicubic spline collocation method for solving numerically the Dirichlet problem for the Poisson's equations (1). Such methods have been developed in many papers in the last decades (see [1], [2], [5], [8], [12], [13], [14]).

2. SPACE OF HERMITE CUBIC SPLINES

Let N be a positive integer and let $\Delta : 0 = t_0 < t_1 < t_2 \cdots < t_N = 1$ be a uniform partition of $[0, 1]$, such that $t_n = n \cdot h$, $n = 0, 1, \ldots, N$, where $h = 1/N$ is the stepsize.

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Let, S_h be the space of Hermite cubic splines on $[0,1]$, defined by

(2) $$S_h := \left\{ v \in C^1[0,1] : v|_{[t_n,t_{n+1}]} \in P_3,\ n = 0,1,...,N-1 \right\},$$

where P_3 denotes the set of polynomials of degree ≤ 3 and let define:

(3) $$S_h^0 := \{ v \in S_h : v(0) = v(1) = 0 \}.$$

G. Fairweather [2] constructed a useful bases for S_h in the following manner. One defines the functions, $v_n,\ s_n \in S_h,\ n = 0,1,...,N$, associated with the point t_n as follows:

$$\begin{aligned} v_n(t_m) &= \delta_{m,n},\quad v_n'(t_m) = 0,\ n,m = 0,1,...,N, \\ s_n(t_m) &= 0,\ s_n'(t_m) = h^{-1}\delta_{m,n},\ n,m = 0,1,...,N, \end{aligned}$$

where $\delta_{n,m}$ is Kronecker symbol.

In order write explicit formulae for v_n and s_n we define the functions:

$$a_1(t) = -2t^3 + 3t^2,\ a_2(t) = t^3 - t^2$$

and the linear mapping $\alpha_n(t) := (t - t_n)/h$ from the interval $[t_n, t_{n+1}]$ onto $[0,1]$. Then we construct the following functions:

$$v_0(t) := \begin{cases} a_1(1 - \alpha_0(t)), & t \in [t_0, t_1] \\ 0, & \text{otherwise} \end{cases}$$

$$v_N(t) = \begin{cases} a_1(\alpha_{N-1}(t)), & t \in [t_{N-1}, t_N] \\ 0, & \text{otherwise} \end{cases}$$

$$v_n = \begin{cases} a_1(\alpha_{N-1}(t)), & t \in [t_{n-1}, t_n], \\ a_1(1 - \alpha_n(t)), & t \in [t_n, t_{n+1}], \quad n = 1,...,N-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$s_0(t) := \begin{cases} -a_2(1 - \alpha_0(t)), & t \in [t_0, t_1] \\ 0, & \text{otherwise} \end{cases}$$

$$s_N(t) = \begin{cases} a_2(\alpha_{N-1}(t)), & t \in [t_{N-1}, t_N] \\ 0, & \text{otherwise} \end{cases}$$

$$s_n(t) = \begin{cases} a_2(\alpha_{n-1}(t)), & t \in [t_{n-1}, t_n] \\ -a_2(1 - \alpha_n(t)), & t \in [t_n, t_{n+1}], \quad n = 1,2,...,N-1 \\ 0, & \text{otherwise.} \end{cases}$$

By ordering v_n and s_n we get two sets of basis functions $\{\Phi_n\}_{n=0}^{2N+1}$ and $\{\Psi_n\}_{n=0}^{2N+1}$ for the spline space S_h:

$$\begin{gathered} \{\Phi_0, \Phi_1, ..., \Phi_{2N}, \Phi_{2N+1}\} := \{\nu_0, v_1, ..., v_{N-1}, s_0, s_1, ..., s_N\}, \\ \{\Psi_0, \Psi_1, ..., \Psi_{2N+1}\} := \{v_0, s_0, v_1, s_1, ..., v_N, s_N\} \end{gathered}$$

and two sets of basis functions for S_h^0

$$\{\Phi_1, ..., \Phi_{2N},\} := \{v_1, ..., v_{N-1}, s_0, s_1, ..., s_N\},$$
$$\{\Psi_1, ..., \Psi_{2N}\} := \{s_0, v_1, s_1, ..., v_{N-1}, s_{N-1}, s_N\}.$$

Let $S_h \otimes S_h$ be the space of Hermite bicubic splines on Ω, that is, the set of all functions on Ω which are finite linear combinations of the functions of the form $u(x)\, v(y)$ where $u, v \in S_h$. Identically, we define the tensorial product space $S_h^0 \otimes S_h^0$ on Ω. Since the dimension of S_h^0 is $2N$, the dimension of $S_h^0 \otimes S_h^0$ is $4N^2$.

Let $\{\xi_m\}_{m=1}^{2N}$ the Gauss points in $]0,1[$ given by:

$$\xi_{2n+1} := t_n + h\tfrac{3-\sqrt{3}}{6}, \quad \xi_{2n+2} := t_n + h\tfrac{3+\sqrt{3}}{6},\ n = 0, 1, ..., N-1$$

and let

$$G := \left\{(x, y) : x, y \in \{\xi_m\}_{m=1}^{2N}\right\} \tag{4}$$

be the set of the Gauss points in Ω.

It is known that each $v \in S_h^0$ is uniquely defined by its values at the Gauss points $\{\xi_m\}_{m=1}^{2N}$. Therefore, in what follows, S_h^0 is regarded as a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ defined by:

$$\langle u, v\rangle := \tfrac{h}{2} \sum_{m=1}^{2N} u(\xi_m)\, v(\xi_m), \quad u, v \in S_h^0. \tag{5}$$

3. HERMITE BICUBIC SPLINE COLLOCATION METHOD

First we consider the homogeneous Dirichlet problem for Poisson's equation on Ω:

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \ \Omega, \\ u &= 0, \quad \text{on } \ \partial\Omega, \end{aligned} \tag{6}$$

where $\Omega := \,]0,1[\, \times \,]0,1[$ and $\partial\Omega$ is the boundary of Ω. Let $\{\Phi_n\}_{n=1}^{2N}$ and $\{\Psi_n\}_{n=1}^{2N}$ be the two bases for S_h^0 above constructed. The piecewise Hermite bicubic spline collocation approximation

$$u_h(x, y) := \sum_{i=1}^{2N} \sum_{j=1}^{2N} u_{i,j} \Phi_i(x)\, \Psi_j(y) \in S_h^0 \otimes S_h^0 \tag{7}$$

to the solution u of (6) is obtained by requiring that:

$$-\Delta u_h(\xi) = f(\xi),\ \xi \in G, \tag{8}$$

where G is defined in (4).

The existence and uniqueness of bicubic spline u_h is proved in [12].

By introducing the vectors:

$$\overline{\mathbf{u}} := [u_{1,1}, u_{1,2}, ...u_{1,2N}, u_{2N,1}, ..., u_{2N,2N}]^{\mathrm{T}}$$

and

$$\overline{\mathbf{f}} := [f_{1,1}, f_{1,2}, ..., f_{1,2N}, ..., f_{2N,1}, ..., f_{2N,2N}]^{\mathrm{T}}, \quad f_{n,m} := f(\xi_n, \xi_m)$$

the system (8) can be written as the system of linear equations

$$(A_\Phi \otimes B_\Psi + B_\Phi \otimes A_\Psi)\,\overline{\mathbf{u}} = \overline{\mathbf{f}}, \tag{9}$$

where the matrices A_Φ and B_Φ, respectively A_Ψ and B_Ψ are defined by:

$$\begin{aligned} A_\Phi &= (a_{m,n})_{m,n=1}^{2N}, \quad a_{m,n} := -\Phi_n''(\xi_m), \\ A_\Psi &= (a_{m,n})_{m,n=1}^{2N}, \quad a_{m,n} := -\Psi_n''(\xi_m), \\ B_\Phi &= (b_{m,n})_{m,n=1}^{2N}, \quad b_{m,n} := \Phi_n(\xi_m), \\ B_\Psi &= (b_{m,n})_{m,n=1}^{2N}, \quad b_{m,n} := \Psi_n(\xi_m), \end{aligned} \tag{10}$$

and in (9) $\otimes$ denotes the matrix tensor product.

It follows from (10) and from construction of the bases of $S_h^0 \otimes S_h^0$ that A_Ψ and B_Ψ are $2N \times 2N$ almost block diagonal matrices with the first and last 2×3 matrix and the others 2×4 blocks in A_Ψ and B_Ψ given by:

$$A_\Psi = h^{-2}\begin{bmatrix} a_1 & a_2 & -a_1 & a_3 \\ -a_1 & -a_3 & a_1 & -a_2 \end{bmatrix}, \quad B_\Psi = \begin{bmatrix} b_1 & b_2 & b_3 & -b_4 \\ b_3 & b_4 & b_1 & -b_2 \end{bmatrix},$$

respectively, where:

$$\begin{aligned} &a_1 = 2\sqrt{3},\ a_2 = 1+\sqrt{3},\ a_3 = \sqrt{3}-1, \\ &b_1 = \tfrac{9+4\sqrt{3}}{18},\ b_2 = \tfrac{3+\sqrt{3}}{36},\ b_3 = \tfrac{9-4\sqrt{3}}{18},\ b_4 = \tfrac{3-\sqrt{3}}{36}. \end{aligned}$$

The first 2×3 block in each matrix is obtained by removing the first column from the 2×4 block of the corresponding matrix, and the last 2×3 block is obtained by removing the third column from 2×4 block of the corresponding matrix.

Because of the special structure of the matrix B_Φ, there are at most four nonzero elements in each its column, and therefore the system (9) can be solved effectively, getting the approximating bicubic spline solution for the problem (6).

Now, we consider the nonhomogeneous Dirichlet problem for the Poisson's equation on Ω:

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \ \Omega, \\ u &= g, \quad \text{on } \ \partial\Omega, \end{aligned} \tag{11}$$

where g is a given function.

The bicubic spline approximating solution $u_h \in S_h^0 \otimes S_h^0$ for (11) is defined in the following way:

$$u_h(x,y) = \sum_{i=0}^{2N+12} \sum_{j=0}^{N+1} u_{ij}\Phi_{ij}(x)\,\Psi_j(y), \tag{12}$$

where the bases $\{\Phi_i\}^{2N+1}$ and $\{\Psi_j\}^{2N+1}$ of the spline space $S_h^0 \otimes S_h^0$ are defined in the previous section.

First we rewrite $u_h(x,y)$ in the form

$$u_h(x,y) = u_h(x,y) + \widetilde{u}_h(x,y), \tag{13}$$

where

$$\overline{u}_h(x,y) := \sum_{i=1}^{2N} \sum_{j=1}^{2N} u_{i,j} \Phi_i(x) \Psi_j(y) \tag{14}$$

corresponds to the bicubic spline collocation solution of the homogeneous Dirichlet problem (6) and

$$\begin{aligned} \widetilde{u}_h(x,y) &= \sum_{j=0}^{2N+1} u_{0,j} \Phi_0(x) \Psi_j(y) + \sum_{j=0}^{2N+1} u_{2N+1,j} \Phi_{2N+1}(x) \Psi_j(y) + \\ &+ \sum_{i=1}^{2N} u_{i,0} \Phi_i(x) \Psi_0(y) + \sum_{i=1}^{2N} u_{i,2N+1} \Phi_i(x) \Psi_{2N+1}(y) \end{aligned} \tag{15}$$

corresponds to the nonhomogeneous boundary condition in (11).

The coefficients of $\widetilde{u}_h$ in (15) can be determined independently therefore the existence and uniqueness of the bicubic spline approximate solution to the nonhomogeneous Dirichlet problem for Poisson's equation (11) follows directly from that of the homogeneous Dirichlet problem.

Following Bialecki and Cai [1] we present two approach to determine the coefficients $\widetilde{u}_h$ in (15) which we refer to as the boundary coefficients of u_h.

In the first approach we approximate the boundary condition $u = g$ using the Hermite cubic spline interpolant of g on each side of $\partial\Omega$. On the left and right hand sides of $\partial\Omega$ we require:

$$(u_h - g)(0, t_n) = 0, \quad \tfrac{\partial}{\partial y}(u_h - g)(0, t_n) = 0, \quad n = 0, 1, ..., N, \tag{16}$$

$$(u_h - g)(1, t_n) = 0, \quad \tfrac{\partial}{\partial y}(u_h - g)(1, t_n) = 0, \quad n = 0, 1, ..., N. \tag{17}$$

Substituting (12) in (16) and (17), for the coefficients $\{u_{0,j}\}_{j=0}^{2N+1}$ of (15) corresponding to the left hand side of $\partial\Omega$ we obtain:

$$\begin{aligned} u_{0,2n} &= g(0, t_n), \quad u_{0,2n+1} = h\tfrac{\partial g}{\partial y}(0, t_n), \quad n = 0, 1, ..., N-1, \\ u_{0,2N} &= h\tfrac{\partial g}{\partial y}(0, t_N), \quad u_{0,2N+1} = g(0, t_N), \end{aligned}$$

and for the coefficients $\{u_{2N+1,j}\}_{j=0}^{2N+1}$ of (15) corresponding to the right hand side of $\partial\Omega$, we have:

$$\begin{aligned} u_{2N+1,2n} &= g(1, t_n), \; u_{2N+1,2n+1} = h\frac{\partial g}{\partial y}(1, t_n), \; n = 0, 1, ..., N-1, \\ u_{2N+1,2N} &= h\frac{\partial g}{\partial y}(1, t_N), \; u_{2N+1,2N+1} = g(1, t_N). \end{aligned}$$

On the bottom and top sides of $\partial\Omega$ we require that

$$(u_h - g)(t_n, 0) = 0, \quad n = 1, ..., N-1, \ \frac{\partial}{\partial x}(u_n - g)(t_n, 0) = 0, \quad n = 0, ..., N,$$
$$(u_h - g)(t_n, 1) = 0, \quad n = 1, ..., N-1, \quad \frac{\partial}{\partial x}(u_h - g)(t_n, 1) = 0, \ n = 0, ..., N.$$

Substituting (12) into above relations we obtain explicitly the coefficients $\{u_{i,0}\}_{i=1}^{2N}$ of (15) corresponding to the bottom side of $\partial\Omega$:

$$u_{n,0} = g(t_n, 0), \quad n = 1, ..., N-1, \quad u_{N+n,0} = h\frac{\partial g}{\partial x}(t_n, 0), \quad n = 0, ..., N,$$

and the coefficients $\{u_{i,2N+1}\}_{i=1}^{2N}$ of (15) corresponding to the top side of $\partial\Omega$:

$$u_{n,2N+1} = g(t_n, 1), \ n = 1, ..., N-1; \quad u_{N+n,2N+1} = h\frac{\partial g}{\partial x}(t_n, 1), \ n = 0, ..., N.$$

In the second approach we approximate $u = g$ on $\partial\Omega$ using the cubic spline interpolant at the boundary Gauss points. Thus, on the left and right hand sides of $\partial\Omega$, we require

$$(u_h - g)(0, 0) = 0, \ (u_h - g)(0, \xi_m) = 0, \ m = 1, ..., 2N; \ (u_h - g)(0, 1) = 0$$

and

$$(u_h - g)(1, 0) = 0, \ (u_h - g)(1, \xi_m) = 0, \ m = 1, ..., 2N; \ (u_h - g)(1, 1) = 0,$$

respectively. By substituting (12) in the first above relation we obtain the following relationships among the coefficients $\{u_{0,j}\}_{j=0}^{2N+1}$ of (15) corresponding to the left hand side of $\partial\Omega$:

$$u_{0,0} = g(0, 0), \ \sum_{j=0}^{2N+1} u_{0,j}\Psi_j(\xi_m) = g(0, \xi_m), \ m = 1, ..., 2N;$$
$$u_{0,2N+1} = g(0, 1). \tag{18}$$

If we set:

$$\mathbf{u}_0 := (u_{0,0}, u_{0,1}, ..., u_{0,2N}, u_{0,2N+1})^{\mathrm{T}};$$
$$g_0 := \{g(0, 0), g(0, \xi_1), ..., g(0, \xi_{2N}), g(0, 1)\}^{\mathrm{T}},$$

then (18) can be written as the $(2N+2)\times(2N+2)$ almost block linear system, of the form

$$\begin{pmatrix} 1 & 0 & & & & & & & & \\ \times & \times & \times & \times & & & & & & \\ \times & \times & \times & \times & & & & & & \\ & & \times & \times & \times & \times & & & & \\ & & \times & \times & \times & \times & & & & \\ & & & & \cdot & \cdot & \cdot & \cdot & & \\ & & & & & & \times & \times & \times & \times \\ & & & & & & \times & \times & \times & \times \\ & & & & & & & & 1 & 0 \end{pmatrix} = \overline{\mathbf{u}}_0 = \overline{\mathbf{g}}_0 \tag{19}$$

with the same 2×4 blocks. Thus the coefficients $\{u_{0,j}\}_{j=0}^{2N+1}$ can be obtained by solving the system (19). By substituting (12) into the second above boundary relation, the coefficients $\{u_{2N+1,j}\}_{j=0}^{2N+1}$ of (15) corresponding to the right hand side of $\partial\Omega$ can be obtained in a similar way.

On the bottom and top sides of $\partial\Omega$ we require:

$$(u_h - g)(0,0) = 0,\ (u_h - g)(\xi_m, 0) = 0,\ m = 1, 2N;\ (u_h - g)(1,0) = 0$$

and

$$(u_h - g)(0,1) = 0,\ (u_h - g)(\xi_m, 1) = 0,\ m = 1, ..., 2N,\ (u_h - g)(1,1) = 0.$$

The first and last above equations give:

$$u_{0,0} = g(0,0)\,,\ u_{2N+1,0} = g(1,0)\,;\ u_{0,2N+1} = g(0,1)\,;\ u_{2N+1,2N+1} = g(1,1)\,.$$

Using the reordering basis functions $\{\Phi_i\}_{i=1}^{2N}$ as the basis functions $\{\Psi_i\}_{i=1}^{2N}$, the coefficients $\{u_{i,0}\}_{i=1}^{2N}$ and $\{u_{i,2N+1}\}_{i=1}^{2N}$ of (15) corresponding to the bottom and top sides of $\partial\Omega$ can also be determined by solving an almost block diagonal system of the form (19). Consequently, $\widetilde{u}_h$ of (13) is determined.

Now we have only to obtain the coefficients of $\overline{u}_h$ in (14).This can be done by requiring that:

$$-\Delta\overline{u}_h(\xi) = \Delta\widetilde{u}_h(\xi) + f(\xi)\,, \quad \xi \in G, \tag{20}$$

where the right hand side is known. The functions $\bar{u}_h$ can be obtained by employing the same algorithm for solving the homogeneous Dirichlet problem (6).

Note that if $g = 0$, then, in both approaches, all of the coefficients in $\widetilde{u}_h$ are zero. But if $g \neq 0$, then, in general, the approximations obtained will be different.

Converting to the estimation of the error and convergence of the given bicubic spline collocation method, Bialecki and Cai [1] have proved that, on nonuniform partitions, the H^1-norm error bounds for the first and second approach are $O(h^3)$ provided that the exact solution u belongs to $H^5(\Omega)$ and $H^5(\Omega) \cap C^4(\Omega)$, respectively.

Dillery [5] improved and extended these results. In particular, it was shown that the H^1-norm error bound for the second approach is $O(h^2)$ under the assumption that $u \in H^6(\Omega)$, then the L^2-norm of the error for each approach is $O(h^6)$.

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