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A REMARK CONCERNING THE PAPER "AN EQUIVALENCE BETWEEN THE CONVERGENCE OF ISHIKAWA, MANN AND PICARD ITERATIONS"*

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Abstract. In this note we show that a result previously obtained by us [An Equivalence Between the Convergences of Ishikawa, Mann and Picard Iterations, Math. Commun., **8**, pp. 15–22, 2003], holds under weaker assumptions.

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1. INTRODUCTION

Let X be a normed space. Let B be a nonempty, convex subset of X. Let $T : B \to B$ be a contraction with constant $L \in (0, 1)$. Let $x_1, u_1 \in B$ be two arbitrary points. We consider the following iteration, see [2]

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n. \tag{1}$$

The sequence $(\alpha_n)_n \subset (0, 1)$ satisfies the following conditions

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{2}$$

Iteration (1) is known as *Mann iteration*. Also, we consider the *Picard iteration*

$$u_{n+1} = T u_n. \tag{3}$$

We recall the following auxiliary result.

LEMMA 1. [1]. *Let $(\rho_n)_n$ be a nonnegative real sequence satisfying*

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n + \mu_n, \tag{4}$$

where $(\lambda_n)_n \subset (0, 1), \sum_{n=1}^{\infty} \lambda_n = \infty$, $\sigma_n > 0, \forall n \geq 1$, $\sigma_n = o(\lambda_n)$, and $\sum_{n=1}^{\infty} \mu_n < \infty$. Then $\lim_{n\to\infty} \rho_n = 0$.

In [3] we have obtained the following result.

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Theorem 2. [3]. *Let X be a normed space, and B a nonempty convex subset of X. Let $T : B \to B$ be a contraction with constant $L \in (0,1)$. Suppose that there exists $x^* \in B$ such that $Tx^* = x^*$, and let $u_1 = x_1 \in B$. If the Picard iteration $(u_n)_n$ given by* (3) *strongly converges to x^*, and $\|u_{n+1} - u_n\| = o(\alpha_n)$; then the Mann sequence $(x_n)_n$ given by* (1) *strongly converges to x^*. Conversely, if the Mann sequence $(x_n)_n$ given by* (1) *strongly converges to x^*, then the Picard iteration $(u_n)_n$ given by* (3) *strongly converges to x^*.*

Proposition 3. *Theorem* 2 *holds without asumption $\|u_{n+1} - u_n\| = o(\alpha_n)$.*

Proof. Suppose that Picard iteration (3) converges. Then from [3, p. 17], we get

$$\begin{aligned} &\|x_{n+1} - u_{n+1}\| \leq \\ &\leq (1 - \alpha_n(1-L)) \|x_n - u_n\| + (1-\alpha_n) \|u_{n+1} - u_n\| . \end{aligned} \tag{5}$$

Also, we have

$$\begin{aligned} \|u_{n+1} - u_n\| = \|Tu_n - Tu_{n-1}\| \leq L \|u_n - u_{n-1}\| \\ \leq L^{n-1} \|u_2 - u_1\| . \end{aligned} \tag{6}$$

Thus, the inequality (5) becomes

$$\|x_{n+1} - u_{n+1}\| \leq (1 - \alpha_n(1-L)) \|x_n - u_n\| + L^{n-1} \|u_2 - u_1\| . \tag{7}$$

Set $\rho_n := \|x_n - u_n\|$, $\lambda_n := \alpha_n(1-L) \in (0,1)$, $\mu_n := L^{n-1}\|u_2 - u_1\|$ and $\sigma_n := 0$, for all $n \in \mathbb{N}$, to obtain (4). It is clear that $\sum_{n=1}^{\infty} L^{n-1} = 1/(1-L) < \infty$. The assumptions of Lemma 1 are fulfilled and consequently we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{8}$$

Finally, we obtain

$$0 \leq \|x_n - x^*\| \leq \|x_n - u_n\| + \|u_n - x^*\| \to 0, \ (n \to \infty).$$

□

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