REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 33 (2004) no. 1, pp. 97–106
ictp.acad.ro/jnaat

ON SOME BIVARIATE INTERPOLATION PROCEDURES

DIMITRIE D. STANCU* and IOANA TAŞCU†

Abstract. In an important paper published in 1966 by the first author [10] was introduced and investigated a very general interpolation formula for univariate functions, which includes, as special cases, the classical interpolation formulae of Lagrange, Newton, Taylor and Hermite.

The purpose of the present paper is to extend that formula to the two-dimensional case. The remainders are expressed by means of partial divided differences and derivatives.

MSC 2000. 41A05, 41A10, 41A63.

Keywords. Bivariate interpolation, divided differences, bivariate polynomial interpolation formulas of Lagrange, Newton, Taylor, Hermite and Biermann.

1. INTRODUCTION

In this paper we start from a general decomposition formula of divided differences defined for a function $f \in C(D)$, $D = [a,b] \times [c,d]$, and for some groups of nodes from the rectangle D. We deduce a general interpolation formula for bivariate functions, corresponding to some general arrays of points. As special cases we obtain several classical types of interpolation polynomials, including Lagrange, Newton, Biermann, Taylor and Hermite.

2. PRELIMINARIES

We first recall some of the principal results obtained in the paper [10]. Then we shall start from an array of $M+1 = p_0 + p_1 + \cdots + p_m + m + 1$ points containing $m+1$ groups of nodes, denoted, by using subscripts and superscripts, by (a_i^k) $i = 0, \ldots, p_k$, $k = 0, \ldots, m$.

Let us use the following explicit notation for this array

$$A = \left\{ \begin{array}{cccc} a_0^0 & a_1^0 & \ldots & a_{p_0}^0 \\ \vdots & \vdots & & \vdots \\ a_0^m & a_1^m & \ldots & a_{p_m}^m \end{array} \right\}. \tag{2.1}$$

We assume that $a \le a_0^k < a_1^k < \cdots < a_{p_k}^k \le b$, $k = 0, \ldots, m$.

The key role in deducing a general interpolation formula corresponding to the function $f \in C[a,b]$ and the points (a_i^k), $i = 0, \ldots, p_k$, $k = 0, \ldots, m$, is the

*"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania, e-mail: ddstancu@math.ubbcluj.ro.

†North University of Baia Mare, Victoriei 76, 430122, Baia Mare, Romania, e-mail: itascu@ubm.ro.

following decomposition formula for divided differences on the distinct nodes a_i^k:

$$[a_0^0, a_1^0, \dots, a_{p_0}^0; a_0^1, a_1^1, \dots, a_{p_1}^1; \dots; a_0^m, a_1^m, \dots, a_{p_m}^m; f] = \tag{2.2}$$
$$= \sum_{k=0}^{m} \left[a_0^k, a_1^k, \dots, a_{p_k}^k; \tfrac{f(t)}{u_k(t)}\right],$$

where

$$u_k(t) = \gamma_0(x)\gamma_1(x)\dots\gamma_{k-1}(x)\gamma_{k+1}(x)\dots\gamma_m(x), \quad u_0(x) = 1,$$

and

$$\gamma_s(x) = (x - a_0^s)(x - a_1^s)\dots(x - a_{p_s}^s), \quad s = 0, \dots, m.$$

The above brackets represent the symbol for divided differences.

If we introduce the node polynomial

$$u(x) = \prod_{s=0}^{m}(x - a_0^s)(x - a_1^s)\dots(x - a_{p_s}^s), \tag{2.3}$$

then we can write: $u_k(x) = u(x)/\gamma_k(x)$.

By using the decomposition formula (2.2) we can obtain the Stancu general interpolation formula

$$f(x) = (S_M f)(x) + (R_M f)(x), \tag{2.4}$$

where, in terms of divided differences, we have

$$(S_M f)(x) = \sum_{k=0}^{m} u_k(x) \sum_{i=0}^{p_k}(x - a_0^k)(x - a_1^k)\dots(x - a_{i-1}^k)(D_i^k f), \tag{2.5}$$

where

$$(D_i^k f) = \left[a_0^k, a_1^k, \dots, a_i^k; \tfrac{f(t)}{u_k(t)}\right].$$

Obviously $(S_M f)(x)$ is a polynomial of degree not exceeding $M = p_0 + p_1 + \dots + p_m + m$.

The remainder of formula (2.4) has the following expression

$$(R_M f)(x) = u(x)(D_{p_0,p_1,\dots,p_m} f)(x),$$

where

$$(D_{p_0,p_1,\dots,p_m} f)(x) = [x, x_0^0, a_1^0, \dots, a_{p_0}^0; \dots; a_0^m, a_1^m, \dots, a_{p_m}^m; f(t)]$$

is the divided difference on all the points from the table A and x.

Now let us present three remarkable special cases of the above approximation formula.

(i) If $p_0 = p_1 = \dots = p_m = 0$ then we have a single column in the array (2.1) and the Stancu approximation formula (2.4) reduces to the Lagrange interpolation formula corresponding to the function f and the nodes $a_0^0, a_0^1, \dots, a_0^m$.

(ii) In the case $m = 0$ then the array A reduces to the nodes from the first row and we obtain the Newton interpolation formula corresponding to the nodes $a_0^0, a_1^0, \dots, a_{p_0}^0$ and the function f.

(iii) If we assume that the nodes from the group $a_0^k, a_1^k, \dots, a_{p_k}^k$ tend to the same value b_k, $k = 0, \dots, m$, then the polynomial (2.5) becomes the Hermite osculatory interpolation polynomial under the form given in 1931 by P. Johansen [3]:

$$(H_M f)(x) = \sum_{k=0}^{m} u_k(x) \sum_{j=0}^{p_k} \tfrac{(x-b_k)^j}{j!} \left(\tfrac{f(t)}{u_k(t)}\right)^{(j)}_{t=b_k},$$

where we use the notations

$$u(x) = \prod_{k=0}^{m} (x - b_k)^{p_k+1}, \quad u_k(x) = \tfrac{u(x)}{(x-b_k)^{p_k+1}}.$$

It should be noticed that this polynomial can be written also under the more explicit form, given in 1948, by W. Simonsen [5]:

$$(H_M f)(x) = \sum_{k=0}^{m} \sum_{j=0}^{p_k} h_{k,j}(x) f^{(j)}(b_k),$$

where the basic osculatory interpolation polynomials $h_{k,j}(x)$ satisfy the relations $h_{k,j}^{(s)}(b_\nu) = 0$ ($\nu \neq k$, $s = 0, \dots, p_\nu$) and $h_{k,j}^{(s)}(b_k) = \delta_j^s$, $s = 0, \dots, p_k$, where δ_j^s is the Kronecker delta.

(iv) In the case $p_k = k$, $k = 0, \dots, m$, the table A from (2.1) leads us to the following triangular array of $m(m+1)/2$ base points

$$T = \begin{cases} a_0^0 \\ a_0^1 & a_1^1 \\ a_0^2 & a_1^2 & a_2^2 \\ \vdots & \vdots & \vdots & \ddots \\ a_0^m & a_1^m & a_2^m & \dots & a_m^m \end{cases}$$

and in the Stancu interpolation polynomial (2.5) we have to replace $p_k = k$ and $u_k(x) = u(x)/\gamma_k(x)$, where

$$\begin{aligned} u(x) =& (x - a_0^0(x - a_0^1)(x - a_1^1) \dots (x - a_0^m)(x - a_1^m) \dots (x - a_m^m), \quad u_0(x) = 1, \\ \gamma_k(x) =& (x - a_0^k)(x - a_1^k) \dots (x - a_k^k). \end{aligned}$$

3. SOME BIVARIATE INTERPOLATION FORMULAS

Let $C(D)$ be the space of all real-valued functions continuous on the rectangle $D = [a, b] \times [c, d]$.

Besides the $M + 1$ distinct points (a_i^k) from the interval $[a, b]$, we consider also the $N + 1 = s_0 + s_1 + \dots + s_n + n + 1$ distinct points (b_j^r), $j = 0, \dots, s_r$, $r = 0, \dots, n$, from the interval $[c, d]$, assuming that $b_0^r < b_1^r < \dots < b_{s_r}^r$,

$r=0,\dots,n$. We denote by B the table of these points

$$B=\left\{\begin{array}{cccc} b_0^0 & b_1^0 & \dots & b_{s_0}^0\\ b_0^1 & b_1^1 & \dots & b_{s_1}^1\\ \vdots & \vdots & & \vdots\\ b_0^r & b_1^r & \dots & b_{s_s}^r\\ \vdots & \vdots & & \vdots\\ b_0^n & b_1^n & \dots & b_{s_n}^n \end{array}\right\}. \tag{3.1}$$

We want to approximate the function $f\in C(D)$ by an interpolation polynomial $(Tf)(x,y)(x,y)$ relative to the grid of nodes from D:

$$G=\{(a_i^k,b_j^r),\ i=0,\dots,p_k,\ j=0,\dots,s_r,\ k=0,\dots,m,\ r=0,\dots,n\}.$$

In order to find the expression of this bivariate interpolation polynomial using these nodes, we first apply formula presented at (2.4), with respect to the first variable and we obtain

$$f(x,y)=(Sf)(x;y)+(Rf)(x;y), \tag{3.2}$$

where

$$\begin{aligned}
(Sf)(x;y)=&\sum_{k=0}^{m}u_k(x)\sum_{i=0}^{p_k}\omega_{0,i-1}(x)(D_i^kf)(y)+(Rf)(x;y),\\
u_k(x)=&\gamma_0(x)\gamma_1(x)\dots\gamma_{k-1}(x)\gamma_{k+1}(x)\dots\gamma_m(x),\\
\gamma_k(x)=&(x-a_0^k)(x-a_1^k)\dots(x-a_{p_k}^k)\\
\omega_{0,i-1}^k(x)=&(x-a_0^k)(x-a_1^k)\dots(x-a_{i-1}^k),\quad \omega_{0,-1}^k(x)=1,\\
(D_i^kf)(y)=&\left[a_0^k,a_1^k,\dots,a_i^k;\tfrac{f(t,y)}{u_k(t)}\right].
\end{aligned}$$

The remainder has the following expression

$$(Rf)(x;y)=u(x)[x,x_0^0,\dots,a_{p_0}^0;\dots;a_0^m,\dots,a_{p_m}^m;f(t,y)],$$

where

$$u(x)=\gamma_0(x)\gamma_1(x)\dots\gamma_m(x)=u_k(x)\gamma_k(x).$$

Then we apply the above result with respect to the variable y and the points b_j^r from the array B given at (3.1). We obtain

$$(Tf)(x,y)=\sum_{k=0}^{m}\sum_{r=0}^{n}u_k(x)v_r(y)\sum_{i=0}^{p_k}\sum_{j=0}^{s_r}\omega_{0,i-1}^k(x)\gamma_{0,j-1}^r(y)(D_{i,j}^{k,r}f), \tag{3.3}$$

where

$$\begin{aligned}
\gamma_{0,j-1}^r(y)=&(y-b_0^r)(y-b_1^r)\dots(y-b_{j-1}^r),\quad \gamma_{0,-1}^r(y)=1,\\
v_r(y)=&\delta_0(y)\delta_1(y)\dots\delta_{r-1}(y)\delta_{r+1}(y)\dots\delta_n(y),\quad v_0(y)=1,\\
\delta_t(y)=&(y-b_0^t)(y-b_1^t)\dots(y-b_{s_t}^t),\quad k=0,\dots,n,\\
v(y)=&v_r(y)\delta_r(y).
\end{aligned}$$

On the other hand we used the bidimensional divided difference

$$D_{i,j}^{k,r} f = \left[\begin{array}{c} a_0^k, a_1^k, \ldots, a_i^k \\ b_0^r, b_1^r, \ldots, b_j^r \end{array} ; \frac{f(t,z)}{u_k(t)v_r(z)} \right].$$

The remainder of the interpolation formula (3.2) has the following expression

$$\begin{aligned}(Rf)(x,y) &= \\ &= u(x)[x, a_0^0, a_1^0, \ldots, a_{p_0}^0; \ldots; a_0^m, a_1^m, \ldots a_{p_m}^m; f(t,y)] \\ &\quad + v(y) \sum_{k=0}^{m} u_k(x) \sum_{i=0}^{p_k} \omega_{0,i-1}(x) \left[\begin{array}{c} a_0^k, a_1^k, \ldots, a_i^k \\ y, b_0^0, \ldots, b_{s_0}^0, \ldots, b_0^n, \ldots, b_{s_n}^n \end{array} ; \frac{f(t,z)}{u_k(t)} \right].\end{aligned}$$

4. INTERPOLATION FORMULAS USING A RECTANGULAR OR A TRIANGULAR GRID OF NODES

A) In the special cases $p_0 = p_1 = \cdots = p_m = 0$, $s_0 = s_1 = \cdots = s_m = 0$ in the tables (2.1) and (3.1) remain only the first columns (a_0^k), $k = 0, \ldots, m$, and (b_0^r), $r = 0, \ldots, n$ and the nodes will be the points $M_{k,r}(a_0^k, b_0^r)$ which are at the intersections of the vertical lines $x = a_0^k$, $k = 0, \ldots, m$, with the horizontal lines $y = b_0^r$, $r = 0, \ldots, n$, in the plane.

In this case the interpolation polynomial (3.3) becomes

$$(L_{m,n}f)(x,y) = \sum_{k=0}^{m} \sum_{r=0}^{n} \frac{u_k(x)v_r(y)}{u_k(a_0^k)v_r(b_0^r)} f(a_0^k, b_0^r), \tag{4.1}$$

where

$$\begin{aligned} u_k(x) &= (x - a_0^0)(x - a_0^1) \ldots (x - a_0^{k-1})(x - a_0^{k+1}) \ldots (x - a_0^m), \\ v_r(y) &= (y - b_0^0)(y - b_0^1) \ldots (y - b_0^{r-1})(y - b_0^{r+1}) \ldots (y - b_0^n). \end{aligned}$$

At (4.1) we have the bivariate Lagrange interpolation polynomial corresponding to the function $f \in C(D)$ and to the nodes $M_{k,r}$ from the rectangle $D = [a,b] \times [c,d]$.

The corresponding remainder of the interpolation formula

$$f(x,y) = (L_{m,n}f)(x,y) + (R_{m,n}f)(x,y) \tag{4.2}$$

can be expressed by means of the nodal polynomials

$$u_m(x) = \prod_{k=0}^{m} (x - a_0^k), \quad v_n(y) = \prod_{r=0}^{n} (y - b_0^r) \tag{4.3}$$

and the partial divided differences, namely

$$\begin{aligned}(R_{m,n}f)(x,y) &= u_m(x)[x, a_0^0, a_0^1, \ldots, a_0^m; f(t,y)] \\ &\quad + v_n(y)[y, b_0^0, b_0^1, \ldots, b_0^n; f(x,z)] \\ &\quad - u_m(x)v_n(y) \left[\begin{array}{c} x, a_0^0, a_0^1, \ldots, a_0^m \\ y, b_0^0, b_0^1, \ldots, b_0^n \end{array} ; f(t,z) \right].\end{aligned}$$

If we now assume that the function f has continuous partial derivatives $f^{(p,q)}(x,y)$ on the rectangle D then this remainder can be expressed in the following form

$$(4.4)\qquad (R_{m,n}f)(x,y) = \tfrac{u_m(x)}{(m+1)!} f^{(m+1,0)}(\xi,y) + \tfrac{v_n(y)}{(n+1)!} f^{(0,n+1)}(x,\eta) - \tfrac{u_m(x)v_n(y)}{(m+1)!(n+1)!} f^{(m+1,n+1)}(\xi,\eta),$$

where $\xi \in (a,b)$ and $\eta \in (c,d)$ are the same in both terms in which they occur.

B) If we assume that $m = n = 0$ then in the tables (2.1) and (3.1) remain only the first rows: $a_0^0, a_1^0, \ldots, a_{p_0}^0$ and $b_0^0, b_1^0, \ldots, b_{s_0}^0$ and the formula (3.3) leads us to the Newton bivariate interpolation formula

$$(4.5)\qquad f(x,y) = (N_{p_0,s_0}f)(x,y) + (R_{p_0,s_0}f)(x,y),$$

where we have

$$(4.6)\qquad (N_{p_0,s_0}f)(x,y) =$$

$$= \sum_{i=0}^{p_0} \sum_{j=0}^{s_0} (x-a_0^0)(x-a_n^0)\ldots(x-a_{i-1}^0)(y-b_0^0)(y-b_1^0)\ldots(y-b_{j-1}^0)(D_{i,j}^0 f)(t,z)$$

and

$$(D_{i,j}^0 f)(x,y) = \begin{bmatrix} a_0^0, a_1^0, \ldots, a_i^0 \\ b_0^0, b_1^0, \ldots, b_j^0 \end{bmatrix} ; f(t,z) \Bigg]$$

is the bidimensional divided difference of the function f on the indicated nodes.

The remainder of the interpolation formula (4.5) has the following expression, in terms of partial divided differences,

$$\begin{aligned}(R_{p_0,s_0}f)(x,y) =& u_{p_0}(x)[x, a_0^0, a_1^0, \ldots, a_{p_0}^0; f(t,y)] \\ &+ v_{s_0}(y)[y, b_0^0, b_1^0, \ldots, b_{s_0}^0; f(x,z)] \\ &- u_{p_0}(x)v_{s_0}(y) \left[\begin{matrix} x, a_0^0, a_1^0, \ldots, a_{p_0}^0 \\ y, b_0^0, b_1^0, \ldots, b_{s_0}^0 \end{matrix} ; f(t,z) \right].\end{aligned}$$

If $f \in C^{p_0,s_0}(D)$ then we can obtain the following estimation for this remainder

$$(4.7)\qquad (R_{p_0,s_0}f)(x,y) = \tfrac{u_{p_0}(x)}{(p_0+1)!} f^{(p_0+1,0)}(\xi,y) + \tfrac{v_{s_0}(y)}{(s_0+1)!} f^{(0,s_0+1)}(x,\eta) - \tfrac{u_{p_0}(x)v_{s_0}(y)}{(p_0+1)!(s_0+1)!} f^{(p_0+1,s_0+1)}(\xi,\eta),$$

where $\xi \in (a,b)$ and $\eta \in (c,d)$.

C) If we use the notations $p_0 = p$, $a_i^0 = x_i$, $b_j^0 = y_j$ and assume that $s_0 = p - i$, $i = 0,\ldots,m$; $j = 0,\ldots,n$, then we arrive from (4.6) at the Biermann interpolation polynomial [1], [9]:

$$(4.8)\quad (B_p f)(x,y) = \sum_{i=0}^{p} \sum_{j=0}^{p-i} (x-x_0)\ldots(x-x_{i-1})(y-y_0)\ldots(y-y_{j-1}) D_{i,j}(f),$$

where

$$D_{i,j}(f) = \left[\begin{array}{c} x_0, x_1, \ldots, x_i \\ y_0, y_1, \ldots, y_j \end{array} ; f(t,z) \right].$$

The Biermann polynomial is of total (global) degree p in x and y and uses a triangular array of base nodes (x_i, y_i), $i = 0, \ldots, p$, $j = 0, \ldots, (p - i)$.

D) When the elements of the rows from the array (2.1) tend respectively to the same values, that is $a_i^k \to c_k$, $i = 0, \ldots, p_k$, $k = 0, \ldots, m$, where $c_0, c_1, \ldots, c_m$ are distinct numbers, while the elements of the rows from the array (3.1) tend also to distinct values, that is $b_j^r \to d_r$, $j = 0, \ldots, s_r$, $r = 0, \ldots, n$, then we can write the nodal polynomials

$$u(x) = \prod_{i=0}^{m} (x - c_i)^{r_i + 1}, \quad v(y) = \prod_{j=0}^{n} (y - d_j)^{s_j + 1}$$

and

$$u_k(x) = u(x)/(x - c_k)^{p_k + 1}, \quad v_r(y) = v(y)/(y - d_r)^{s_r + 1}.$$

Because we have

$$\omega_{0,i-1}^k(x) = (x - x_k)^i, \quad \gamma_{j-1}^r(y) = (y - d_r)^j,$$

$$\left[\begin{array}{c} a_0^k, a_1^k, \ldots, a_i^k \\ b_0^r, b_1^r, \ldots, b_j^r \end{array} ; \frac{f(t,z)}{u_k(t)v_r(z)} \right] = \left[\begin{array}{c} c_k, c_k, \ldots, c_k \\ d_r, d_r, \ldots, d_r \end{array} ; \frac{f(t,z)}{u_k(t)v_r(z)} \right] = \frac{1}{i!j!} \left(\frac{f(t,z)}{u_k(t)v_r(z)} \right)_{c_k, d_r}^{(i,j)},$$

it follows that the polynomial (3.3) may be expressed in the following form (4.9)

$$(H_{M,N}f)(x,y) = \sum_{k=0}^{m} \sum_{r=0}^{n} u_k(x) v_r(y) \sum_{i=0}^{p_k} \sum_{j=0}^{s_r} \frac{(x-c_k)^i (y-d_r)^j}{i!j!} \left(\frac{f(t,z)}{u_k(t)v_r(z)} \right)_{c_k, d_r}^{(i,j)},$$

which is the Hermite osculatory bivariate interpolation polynomial of degree (M, N), where $M = p_0 + p_1 + \cdots + p_m + m$ and $N = s_0 + s_1 + \cdots + s_n + n$, which enjoys the following properties:

$$(H_{M,N}^{(\nu,\mu)})(c_k, d_r) = f^{(\nu,\mu)}(c_k, d_r),$$

where $\nu = 0, \ldots, p_k$, $\mu = 0, \ldots, s_r$, $k = 0, \ldots, m$ and $r = 0, \ldots, n$.

The remainder of the approximation formula of the function f by this interpolation polynomial has the following expression

$$(R_{M,N}f)(x,y) =$$
$$= u(x)\left[\begin{matrix} x \\ 1 \end{matrix}, \begin{matrix} c_0 \\ p_0+1 \end{matrix}, \begin{matrix} c_1 \\ p_1+1 \end{matrix}, \ldots, \begin{matrix} c_m \\ p_m+1 \end{matrix}; f(t,y)\right]$$
$$+ v(y)\left[\begin{matrix} y \\ 1 \end{matrix}, \begin{matrix} d_0 \\ s_0+1 \end{matrix}, \begin{matrix} d_1 \\ s_1+1 \end{matrix}, \ldots, \begin{matrix} d_n \\ s_n+1 \end{matrix}; f(x,z)\right]$$
$$- u(x)v(y)\cdot$$
$$\cdot\left[\begin{matrix} x \\ 1 \end{matrix}, \begin{matrix} c_0 \\ p_0+1 \end{matrix}, \begin{matrix} c_1 \\ p_1+1 \end{matrix}, \ldots, \begin{matrix} c_m \\ p_m+1 \end{matrix}; \begin{matrix} d_0 \\ s_0+1 \end{matrix}, \begin{matrix} d_1 \\ s_1+1 \end{matrix}, \ldots, \begin{matrix} d_n \\ s_n+1 \end{matrix}; f(t,z)\right]$$

In the brackets, in the first row, there are the coordinates of the nodes and in the second are indicated their corresponding orders of multiplicities.

Using the partial derivatives of the function f, this remainder can be estimated by the following formula

$$\begin{aligned}(R_{M,N}f)(x,y) =& \tfrac{u(x)}{(M+1)!} f^{(M+1,0)}(\xi,y) + \tfrac{v(y)}{(N+1)!} f^{(0,N+1)}(x,\eta) \\ &- \tfrac{u(x)v(y)}{(M+1)!(N+1)!} f^{(M+1,N+1)}(\xi,\eta),\end{aligned}$$

where $\xi \in (a,b)$, $\eta \in (c,d)$.

Let us now mention that the Hermite interpolation polynomial (4.9) can be also expressed in a more explicit form

$$(H_{M,N}f)(x,y) = \sum_{k=0}^{m}\sum_{r=0}^{n}\sum_{i=0}^{p_k}\sum_{j=0}^{s_r} g_{k,i}(x)h_{r,j}(y)f^{(i,j)}(c_k,d_r),$$

where

$$g_{k,i}(x) = \tfrac{(x-c_k)^i}{i!}\Big[\sum_{\nu=0}^{p_k-i} \tfrac{(x-c_k)^\nu}{\nu!}\big(\tfrac{1}{u_k(t)}\big)^{(\nu)}_{c_k}\Big]u_k(x)$$

and

$$h_{r,j}(y) = \tfrac{(y-d_r)^j}{j!}\Big[\sum_{r=0}^{s_r-j} \tfrac{(y-d_r)^\mu}{\mu!}\big(\tfrac{1}{v_r(z)}\big)^{(\mu)}_{d_r}\Big]v_r(y).$$

In the particular case $m = n = 0$, $p_0 = p$, $s_0 = s$ we obtain the Taylor-type bivariate formula

$$f(x,y) = \sum_{k=0}^{p}\sum_{r=0}^{s} \tfrac{(x-c_0)^k(y-d_0)^r}{k!r!} f^{(k,r)}(c_0,d_0) + (Rf)(x,y), \tag{4.10}$$

where

$$\begin{aligned}(Rf)(x,y) =& \tfrac{(x-c_0)^{p+1}}{(p+1)!} f^{(p+1,0)}(\xi,y) + \tfrac{(y-d_0)^{s+1}}{(s+1)!} f^{(0,s+1)}(x,\eta) \\ &- \tfrac{(x-c_0)^{p+1}(y-d_0)^{s+1}}{(p+1)!(s+1)!} f^{(p+1,s+1)}(\xi,\eta),\end{aligned}$$

ξ and η being the same in the terms in which they occur.

By using the integration by parts, the first author has obtained in [8] the following integral representation for the remainder of the Taylor-type formula (4.10)

$$(Rf)(x,y)=\int_{c_0}^{x}\tfrac{(x-t)^p}{p!}f^{(p+1,0)}(t,y)\mathrm{d}t+\int_{d_0}^{y}\tfrac{(y-z)^s}{s!}f^{(0,s+1)}(x,z)\mathrm{d}z$$
$$-\int_{c_0}^{x}\int_{d_0}^{y}\tfrac{(x-t)^p(y-z)^s}{p!s!}f^{(p+1,s+1)}(t,z)\mathrm{d}t\mathrm{d}z.$$

It should be further noted that employing the Biermann interpolation polynomial given at (4.8) we can obtain as a limit case the Taylor bivariate polynomial of total degree m:

$$(T_pf)(x,y)=\sum_{i=0}^{p}\sum_{j=0}^{p-i}\tfrac{(x-c)^i(y-d)^j}{i!j!}f^{(i,j)}(c,d). \tag{4.11}$$

If we assume that f belongs to the class C^{p+1} of functions having continuous all the partial derivatives of orders $(p+1-i,i)$, $(i=0,1,\dots,p+1)$ in a neighborhood $E_{c,d}$ of the point (c,d), then the remainder R_pf of the approximation formula of f by the bivariate Taylor polynomial (4.11) can be represented under the following form

$$(R_pf)(x,y)=$$
$$=\tfrac{1}{p!}\int_0^1(1-u)^n\big[(x-c)\tfrac{\partial}{\partial x}+(y-d)\tfrac{\partial}{\partial y}\big]^{(p+1)}f(c+(x-c)u,d+(y-d)u)\mathrm{d}u,$$

whenever the point (x,y) belongs to $E_{c,d}$. This formula was deduced in the paper [8] of the first author.

REFERENCES

[1] Biermann, O., *Über näherungsweise Kubaturen*, Monatsh. Math. Phys., **14**, pp. 211–225, 1903.

[2] Hermite, C., *Sur la formule d'interpolation de Lagrange*, J. Reine Angew. Math., **84**, pp. 70–79, 1878.

[3] Johansen, P., *Über osculierende Interpolation*, Skand. Actuarietidskr, **14**, pp. 231–237, 1931.

[4] Salzer, A., *A multi-point generalization of Newton's divided difference formula*, Proc. Amer. Math. Soc., **13**, pp. 210–212, 1962.

[5] Simonsen, W., *On divided differences and osculatory interpolation*, Skand. Akturietidskr., **31**, pp. 157–164, 1948.

[6] Stancu, D. D., *On the interpolation formula of Hermite and on some applications of it* (Romanian), Acad. R. P. Rom. Fil. Cluj, Stud. Cerc. Mat., **8**, pp. 339–355, 1957.

[7] Stancu, D. D., *Considerations on the polynomial interpolation formulas for functions of several variables*, Bul. Univ. Babeş-Bolyai, Ser. St. Nat. (Romanian), **1**, pp. 43–82, 1957.

[8] Stancu, D. D., *On the integral representation of the remainder in a Taylor formula of two variables*, Acad. R. P. Rom. Fil. Cluj, Stud. Cerc. Mat., **13**, pp. 175–182, 1962.

[9] Stancu, D. D., *The remainder of certain linear approximation formulas in two variables*, J. SIAM Numer. Anal. Ser. B, **1**, pp. 137–143, 1964.

[10] Stancu, D. D., *On Hermite's osculatory interpolation formula and on some generalizations of it*, Mathematica (Cluj), **8**(**31**), pp. 373–391, 1966.

Received by the editors: January 15, 2004.