THE VORONOVSKAJA THEOREM
FOR BERNSTEIN-SCHURER BIVARIATE OPERATORS

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Abstract. The Voronovskaja theorem for the Bernstein–Schurer bivariate operators is established.

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1. PRELIMINARIES

For the classical Bernstein operator $B_m : C([0, 1]) \to C([0, 1])$, defined for any function $f \in C([0, 1])$ by

$$ (B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) \, f\left(\frac{k}{m}\right), $$

where

$$ p_{m,k} = \binom{m}{k} x^k (1-x)^{m-k}, $$

Voronovskaja E., proved (see [6]) the result expressed in the following result.

Theorem 1. Let $f \in C([0, 1])$ be a function two times derivable in the point $x \in [0, 1]$. Then, the equality

$$ \lim_{m \to \infty} m\{(B_m f)(x) - f(x)\} = \frac{x(1-x)}{2} \, f''(x) $$

holds.

The above result is known as “the Voronovskaja theorem”.

Let $p$ be a non-negative integer. In 1962, Schurer F. (see [5]) introduced and studied the operator $\tilde{B}_{m,p} : C([0, 1 + p]) \to C([0, 1])$, defined for any function $f \in C([0, 1 + p])$ by:

$$ (\tilde{B}_{m,p} f)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) \, f\left(\frac{k}{m}\right), $$

where

$$ \tilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}. $$

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In our earlier paper [2] was proved the following Voronovskaja-type theorem for the operator (4).

**Theorem 2.** [2, Th. 2.1]. Let \( p \) be a non-negative integer. If the function \( f \in C([0,1+p]) \) is two times derivable in the point \( x \in [0,1+p] \), the following equality

\[
\lim_{m \to \infty} (m+p) \left\{ (\tilde{B}_{m,p} f)(x) - f(x) \right\} = p \cdot f'(x) + \frac{x(1-x)}{2} f''(x)
\]

holds.

Let \( p,q \) be two non-negative integers. Using the method of parametric extensions in the paper [3] we construct the bivariate operator of Bernstein–Schurer \( \tilde{B}_{m,n,p,q} : C([0,1+p] \times [0,1+q]) \to C([0,1] \times [0,1]) \), defined for any function \( f \in C([0,1+p] \times [0,1+q]) \) by

\[
(\tilde{B}_{m,n,p,q} f)(x,y) = \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \tilde{p}_{m,k}(x)\tilde{p}_{n,j}(y) \ f\left(\frac{k}{m},\frac{j}{n}\right).
\]

We proved (see [3, Th. 2.4]) that the sequence \( \{\tilde{B}_{m,n,p,q} f\}_{m,n \in \mathbb{N}} \) converges to \( f \), uniformly on \([0,1+p] \times [0,1+q]\), for any \( f \in C([0,1+p] \times [0,1+q]) \). An estimation of the approximation order of \( f \in C^2([0,1+p] \times [0,1+q]) \) by \( \tilde{B}_{m,n,p,q} f \) was also gived, using the first order modulus of smoothness for bivariate functions (see [3, Th. 2.5]).

The aim of the present paper is to establish a Voronovskaja-type theorem for the bivariate Bernstein–Schurer operators, i.e. to calculate the following limit

\[
\lim_{m \to \infty} \left\{ (\tilde{B}_{m,m,p,q} f)(x,y) - f(x,y) \right\},
\]

where \( f \in C^{2,2}([0,1+p] \times [0,1+q]) \).

**2. Auxiliary Result**

We have to give some properties of the \( \tilde{B}_{m,p} \) and \( \tilde{B}_{m,n,p,q} \) operators.

**Lemma 3.** [1]. For any fixed point \( x \in [0,1+p] \), there exists a positive constant \( M_1(x,p) \), depending on \( x \) and \( p \), such that

\[
\tilde{B}_{m,p}((t-x)^4;x) \leq M_1(x,p) \ n^{-2}
\]

for all \( x \in \mathbb{N}^* \).

**Lemma 4.** [3]. If \((x,y) \in [0,1+p] \times [0,1+q]\) and \( m,n \in \mathbb{N}^* \) the following equalities hold

(i) \( \tilde{B}_{m,n,p,q} (e_{00};x,y) = 1; \)

(ii) \( \tilde{B}_{m,n,p,q} (e_{10};x,y) = \{1+pm^{-1}\}x; \)

(iii) \( \tilde{B}_{m,n,p,q} (e_{01};x,y) = \{1+qn^{-1}\}y; \)
(iv) \( \hat{B}_{m,n,p,q}(e_{22}; x, y) = (m+p)m^{-2}\{(m+p)x^2 + x(1-x)\} + \\
+ (n+q)n^{-2}\{(n+q)y^2 + y(1-y)\}, \)
where \( e_{00}(t, \tau) = 1, \) \( e_{10}(t, \tau) = t, \) \( e_{01}(t, \tau) = \tau, \) \( e_{22}(t, \tau) = t^2 + \tau^2 \) are the test functions.

**Lemma 5.** Let \((x_0, y_0) \in [0, 1 + p] \times [0, 1 + q]\) be a fixed point and let \( \varphi \in C([0, 1 + p] \times [0, 1 + q]) \) such that \( \varphi(x_0, y_0) = 0. \) Then

\[
\lim_{n \to \infty} \hat{B}_{m,n,p,q}(\varphi; x_0, y_0) = 0.
\]

**Proof.** By the properties of function \( \varphi, \) for every \( \varepsilon > 0 \) there exists a positive constant \( \delta = \delta(\varepsilon) \) such that for \( |t - x_0| < \delta, \) \( |	au - y_0| < \delta \) the following holds

\[
|\varphi(t, \tau)| < \frac{\varepsilon}{10}.
\]

On the other hand, because \( f \in C([0, 1 + p] \times [0, 1 + q]), \) there exists a positive constant \( C_1, \) such that

\[
|\varphi(t, \tau)| \leq C_1.
\]

Next, we can write:

\[
|\hat{B}_{m,n,p,q}(\varphi; x_0, y_0)| \leq \sum_{k=0}^{m+p} \sum_{j=0}^{m+q} \hat{p}_{m,k}(x_0) \hat{p}_{m,j}(y_0) |\varphi(\frac{k}{m}, \frac{j}{m})|.
\]

Let us to divide the set of sum’s indices in the following four classes

\[
I_1 = \{ (k, j) : |k/m - x_0| < \delta \text{ and } |j/m - y_0| < \delta \}; \\
I_2 = \{ (k, j) : |k/m - x_0| < \delta \text{ and } |j/m - y_0| \geq \delta \}; \\
I_3 = \{ (k, j) : |k/m - x_0| \geq \delta \text{ and } |j/m - y_0| < \delta \}; \\
I_4 = \{ (k, j) : |k/m - x_0| \geq \delta \text{ and } |j/m - y_0| \geq \delta \}.
\]

If we denote \( \omega_{k,j}(x_0, y_0) = \hat{p}_{m,k}(x) \hat{p}_{m,j}(y)|\varphi(\frac{k}{m}, \frac{j}{m})|, \) we can write:

\[
|\hat{B}_{m,n,p,q}(\varphi; x_0, y_0)| \leq \sum_{(k,j) \in I_1} \omega_{k,j}(x_0, y_0) + \sum_{(k,j) \in I_2} \omega_{k,j}(x_0, y_0) + \\
\sum_{(k,j) \in I_3} \omega_{k,j}(x_0, y_0) + \sum_{(k,j) \in I_4} \omega_{k,j}(x_0, y_0).
\]

Let \( S_1, S_2, S_3, S_4 \) be the sums of the right side of (12). For the sum \( S_1, \) we get that there exists a natural number \( m_1 = m_1(\varepsilon) \) such that

\[
S_1 = \sum_{(k,j) \in I_1} \omega_{j,k}(x_0, y_0) \leq \sum_{k=0}^{m+p} \sum_{j=0}^{m+p} \hat{p}_{m,k}(x_0) \hat{p}_{m,j}(y_0) |\varphi(\frac{k}{m}, \frac{j}{m})| \\
< \frac{\varepsilon}{4} \hat{B}_{m,n,p,q}(e_{00}; x_0, y_0) < \frac{\varepsilon}{4}
\]
for \( m > m_1. \)
Because $|\frac{j}{m} - y_0| \geq \delta$ implies $\delta^{-2}(\frac{j}{m} - y_0)^2 \geq 1$, for the sum $S_2$, we get

$$S_2 = \sum_{(k,j) \in I_2} \omega_{j,k}(x_0, y_0)$$

$$\leq \delta^{-2} \sum_{(k,j) \in I_2} \left( \frac{j}{m} - y_0 \right)^2 \tilde{p}_{m,k}(x_0) \tilde{p}_{m,j}(y_0) |\varphi(k, \frac{j}{m})|$$

$$\leq M \cdot \delta^{-2} \sum_{(k,j) \in I_2} \left( \frac{j}{m} - y_0 \right)^2 \tilde{p}_{m,k}(x_0) \tilde{p}_{m,j}(y_0) \leq$$

$$\leq M \cdot \delta^{-2} \sum_{m \geq 0} \sum_{j=0}^{m} \left( \frac{j}{m} - y_0 \right)^2 \tilde{p}_{m,k}(x_0) \tilde{p}_{m,j}(y_0)$$

$$= M \cdot \delta^{-2} \tilde{B}_{m,p,q}(\tau - y_0)^2; x_0, y_0),$$

where $M = \max \{ |\varphi(x, y)| : (x, y) \in [0, 1 + p] \times [0, 1 + q] \}$.

Taking into account Lemma 3, we get that there exists a positive constant $M_2(m, q, y_0)$ such that

$$S_2 \leq M_2(m, q, y_0) y_0 (1 - y_0) \cdot m^{-1} \delta^{-2}.$$

In the same way, applying Lemma 3 and Lemma 4 we obtain

$$S_3 \leq M_3(m, q, x_0) x_0 (1 - x_0) \cdot m^{-1} \delta^{-2}$$

and

$$S_4 \leq M_4(m, p, q, x_0, y_0) x_0 y_0 (1 - x_0)(1 - y_0) \cdot m^{-2} \delta^{-4}.$$

It follows that for given positive numbers $\varepsilon$, $\delta$, $M_i$ ($i = 1, 2, 3$) and $(x_0, y_0) \in [0, 1 + p] \times [0, 1 + q]$ there exists the natural numbers $m_2$, $m_3$, $m_4$ such that

$$S_2 \leq M_2(m, q, y_0) y_0 (1 - y_0) \cdot m^{-1} \delta^{-2} < \frac{\varepsilon}{4}, \text{ for } m > m_2,$$

$$S_3 \leq M_3(m, q, x_0) x_0 (1 - x_0) \cdot m^{-1} \delta^{-2} < \frac{\varepsilon}{4}, \text{ for } m > m_3,$$

$$S_4 \leq M_4(m, p, q, x_0, y_0) x_0 y_0 (1 - x_0)(1 - y_0) \cdot m^{-2} \delta^{-4} < \frac{\varepsilon}{4}, \text{ for } m > m_4.$$

Therefore, there exists the natural number $N = \max\{m_1, m_2, m_3, m_4\}$ such that for all natural numbers $n > N$ we have $S_1 + S_2 + S_3 + S_4 < \varepsilon$.

We can conclude that $\lim_{m \to \infty} \tilde{B}_{m,p,q}(\varphi; x_0, y_0) = 0$, and the proof ends. 

3. THE MAIN RESULT

Using the auxiliary results from section 2, we can now prove the Voronovskaja type theorem for the bivariate operators of Bernstein–Schurer.

Let us to denote by $C^{2,2}([0, 1 + p] \times [0, 1 + q])$ the space of all functions $f \in C ([0, 1 + p] \times [0, 1 + q])$ with partial derivatives of first and second order belonging to the space $C ([0, 1 + p] \times [0, 1 + q])$. 
Theorem 6. Let \( f \in C^{2,2}([0,1+p] \times [0,1+q]) \). Then, for all \((x,y) \in [0,1+p] \times [0,1+q]\) we have

\[
\lim_{m \to \infty} m \left\{ \tilde{B}_{m,m,p,q}(f; x, y) - f(x, y) \right\} = p f'_x(x, y) + q f'_y(x, y) + \frac{1}{2} \left\{ x(1-x) f''_{xx}(x, y) + y(1-y) f''_{yy}(x, y) \right\}.
\]

Proof. Let \((x_0, y_0) \in [0,1+p] \times [0,1+q]\) be a fixed point. By making use of the Taylor formula for \((t, \tau) \in [0,1+p] \times [0,1+q]\) with the Peano’s form of the remainder term, we have

\[
f(t, \tau) = f(x_0, y_0) + (t-x) \cdot f'_x(x_0, y_0) + (\tau-y) \cdot f'_y(x_0, y_0) + \frac{1}{2} \left\{ (t-x)^2 f''_{xx}(x_0, y_0) + 2(t-x)(\tau-y) f''_{xy}(x_0, y_0) + (\tau-y)^2 f''_{yy}(x_0, y_0) \right\} + \varphi(t, \tau; x_0, y_0) \sqrt{(t-x)^4 + (\tau-y)^4},
\]

where \(\varphi(\cdot, \cdot; x_0, y_0) \in C([0,1+p] \times [0,1+q])\) and \(\lim_{t \to x_0, \tau \to y_0} \varphi(t, \tau; x_0, y_0) = 0\).

Taking into account Lemma 3 we get

\[
\tilde{B}_{m,m,p,q}(f(t, \tau); x_0, y_0) - f(x_0, y_0) = p \cdot m^{-1} x f'_x(x_0, y_0) + q m^{-1} y f'_y(x_0, y_0) + \frac{1}{2} \left\{ (p^2 m^{-2} x^2_0 + (m+p)m^{-2} x_0(1-x_0)) f''_{xx}(x_0, y_0) + 2pqm^{-2} x_0 y_0 f''_{xy}(x_0, y_0) + q^2 m^{-2} y^2 + (m+q) m^{-2} y_0(1-y_0) f''_{yy}(x_0, y_0) \right\} + \tilde{B}_{m,m,p,q} \left( \varphi(t, \tau; x_0, y_0) \sqrt{(t-x)^4 + (\tau-y)^4}; x_0, y_0 \right).
\]

By Cauchy’s inequality, it follows

\[
\left| \tilde{B}_{m,m,p,q} \left( \varphi(t, \tau; x_0, y_0) \sqrt{(t-x)^4 + (\tau-y)^4}; x_0, y_0 \right) \right| \leq \left\{ \tilde{B}_{m,m,p,q} \left( \varphi^2(t, \tau; x_0, y_0) \right) \right\}^{\frac{1}{2}} \left\{ \tilde{B}_{m,m,p,q} \left( (t-x)^4 + (\tau-y)^4; x_0, y_0 \right) \right\}^{\frac{1}{2}}.
\]

From Lemma 4 it follows

\[
\lim_{m \to \infty} \tilde{B}_{m,m,p,q}(\varphi^2(t, \tau; x_0, y_0); x_0, y_0) = 0
\]

and from Lemma 4 we get

\[
\tilde{B}_{m,m,p,q} \left( (t-x)^4 + (\tau-y)^4; x_0, y_0 \right) = \tilde{B}_{m,p}((t-x)^4; x_0) + \tilde{B}_{m,q}((\tau-y)^4; y_0) \leq \{ M_1(x_0, p) + M_2(y_0, q) \} m^{-2},
\]

where \(M_1(x_0, p)\) and \(M_2(y_0, q)\) are positive constants. Taking into account (15) and (16), by (14) we arrive to the desired result. \(\square\)
REFERENCES


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