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THE VORONOVSKAJA THEOREM FOR BERNSTEIN-SCHURER BIVARIATE OPERATORS

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Abstract. The Voronovskaja theorem for the Bernstein–Schurer bivariate operatos is established.

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1. PRELIMINARIES

For the classical Bernstein operator $B_m : C([0,1]) \to C([0,1])$, defined for any function $f \in C([0,1])$ by

$$(B_m\, f)(x) = \sum_{k=0}^{m} p_{m,k}(x)\ f\left(\tfrac{k}{m}\right), \tag{1}$$

where

$$p_{m,k} = \tbinom{m}{k}\, x^k\, (1-x)^{m-k}, \tag{2}$$

Voronovskaja E., proved (see [6]) the result expressed in the following result.

THEOREM 1. *Let $f \in C([0,1])$ be a function two times derivable in the point $x \in [0,1]$. Then, the equality*

$$\lim_{m\to\infty} m\{(B_m\ f)(x) - f(x)\} = \tfrac{x(1-x)}{2}\ f''(x) \tag{3}$$

holds.

The above result is known as "the Voronovskaja theorem".

Let p be a non-negative integer. In 1962, Schurer F. (see [5]) introduced and studied the operator$\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, defined for any function $f \in C([0,1+p])$ by:

$$(\widetilde{B}_{m,p}\, f)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x)\, f\left(\tfrac{k}{m}\right), \tag{4}$$

where

$$\widetilde{p}_{m,k}(x) = \tbinom{m+p}{k} x^k (1-x)^{m+p-k}. \tag{5}$$

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In our earlier paper [2] was proved the following Voronovskaja-type theorem for the operator (4).

THEOREM 2. [2, Th. 2.1]. *Let p be a non-negative integer. If the function $f \in C([0,1+p])$ is two times derivable in the point $x \in [0,1+p]$, the following equality*

$$\lim_{m\to\infty}(m+p)\left\{(\widetilde{B}_{m,p}\ f\ )(x)-f(x)\right\}=p\cdot f'(x)+\tfrac{x(1-x)}{2}\ f''(x) \tag{6}$$

holds.

Let p, q be two non-negative integers. Using the method of parametric extensions in the paper [3] we construct the bivariate operator of Bernstein–Schurer $\widetilde{B}_{m,n,p,q} : C([0,1+p]\times[0,1+q]) \to C([0,1]\times[0,1])$, defined for any function $f \in C([0,1+p]\times[0,1+q])$ by

$$(\widetilde{B}_{m,n,p,q}\ f)(x,y)=\sum_{k=0}^{m+p}\sum_{j=0}^{n+q}\widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y)\ f\left(\tfrac{k}{m},\tfrac{j}{n}\right). \tag{7}$$

We proved (see [3, Th. 2.4]) that the sequence $\{\widetilde{B}_{m,n,p,q}\ f\}_{m,n\in\mathbb{N}}$ converges to f, uniformly on $[0,1+p]\times[0,1+q]$, for any $f \in C([0,1+p]\times[0,1+q])$. An estimation of the approximation order of $f \in C([0,1+p]\times[0,1+q])$ by $\widetilde{B}_{m,n,p,q}\ f$ was also gived, using the first order modulus of smoothness for bivariate functions (see [3, Th. 2.5]).

The aim of the present paper is to establish a Voronovskaja-type theorem for the bivariate Bernstein–Schurer operators, i.e. to calculate the following limit

$$\lim_{m\to\infty}\left\{(\widetilde{B}_{m,m,p,q}\ f)(x,y)-f(x,y)\right\}, \tag{8}$$

where $f \in C^{2,2}([0,1+p]\times[0,1+q])$.

2. AUXILLIARY RESULT

We have to give some properties of the $\widetilde{B}_{m,p}$ and $\widetilde{B}_{m,n,p,q}$ operators.

LEMMA 3. [1]. *For any fixed point $x \in [0,1+p]$, there exists a positive constant $M_1(x,p)$, depending on x and p, such that*

$$\widetilde{B}_{m,p}((t-x)^4;x) \le M_1(x,p)\,n^{-2}$$

for all $x \in \mathbb{N}^*$.

LEMMA 4. [3]. *If $(x,y) \in [0,1+p]\times[0,1+q]$ and $m,n \in \mathbb{N}^*$ the following equalities hold*

(i) $\widetilde{B}_{m,n,p,q}\,(e_{00};x,y)=1$;
(ii) $\widetilde{B}_{m,n,p,q}\,(e_{10};x,y)=\{1+p\,m^{-1}\}x$;
(iii) $\widetilde{B}_{m,n,p,q}\,(e_{01};x,y)=\{1+q\,n^{-1}\}y$;

(iv) $\widetilde{B}_{m,n,p,q}\,(e_{22};x,y) = (m+p)\,m^{-2}\{(m+p)x^2 + x(1-x)\} +$
$+(n+q)\,n^{-2}\{(n+q)y^2 + y(1-y)\},$

where $e_{00}(t,\tau)=1,\ e_{10}(t,\tau)=t,\ e_{01}(t,\tau)=\tau,\ e_{22}(t,\tau)=t^2+\tau^2$ *are the test functions.*

LEMMA 5. *Let* $(x_0,y_0)\in[0,1+p]\times[0,1+q]$ *be a fixed point and let* $\varphi\in C\left([0,1+p]\times[0,1+q]\right)$ *such that* $\varphi(x_0,y_0)=0$. *Then*

$$\lim_{n\to\infty}\widetilde{B}_{m,m,p,q}(\varphi;x_0,y_0)=0.$$

Proof. By the properties of function φ, for every $\varepsilon>0$ there exists a positive constant $\delta=\delta(\varepsilon)$ such that for $|t-x_0|<\delta\,,\ |\tau-y_0|<\delta$ the following holds

$$|\varphi(t,\tau)|<\tfrac{\varepsilon}{4}. \tag{9}$$

On the other hand, because $f\in C\left([0,1+p]\times[0,1+q]\right)$, there exists a positive constant C_1, such that

$$|\varphi(t,\tau)|\le C_1. \tag{10}$$

Next, we can write:

$$\left|\widetilde{B}_{m,m,p,q}(\varphi;\,x_0,y_0)\right|\le\sum_{k=0}^{m+p}\ \sum_{j=0}^{m+q}\widetilde{p}_{m,k}(x_0)\widetilde{p}_{m,j}(y_0)\left|\varphi\left(\tfrac{k}{m},\tfrac{j}{m}\right)\right|. \tag{11}$$

Let us to divide the set of sum's indices in the following four classes

$$\begin{aligned}
I_1&=\{(k,j):|k/m-x_0|<\delta \text{ and } |j/m-y_0|<\delta\}\,;\\
I_2&=\{(k,j):|k/m-x_0|<\delta \text{ and } |j/m-y_0|\ge\delta\}\,;\\
I_3&=\{(k,j):|k/m-x_0|\ge\delta \text{ and } |j/m-y_0|<\delta\}\,;\\
I_4&=\{(k,j):|k/m-x_0|\ge\delta \text{ and } |j/m-y_0|\ge\delta\}\,.
\end{aligned}$$

If we denote $\omega_{k,j}(x_0,y_0)=\widetilde{p}_{m,k}(x)\widetilde{p}_{m,j}(y)\left|\varphi\left(\frac{k}{m},\frac{j}{m}\right)\right|$, we can write:

$$\begin{aligned}
\left|\widetilde{B}_{m,m,p,q}(\varphi;\,x_0,y_0)\right| \quad\le\quad &\sum_{(k,j)\in I_1}\omega_{k,j}(x_0,y_0)+\sum_{(k,j)\in I_2}\omega_{k,j}(x_0,y_0)\\
&+\sum_{(k,j)\in I_3}\omega_{k,j}(x_0,y_0)+\sum_{(k,j)\in I_4}\omega_{k,j}(x_0,y_0).
\end{aligned} \tag{12}$$

Let $S_1,\ S_2,\ S_3,\ S_4$ be the sums of the right side of (12). For the sum S_1, we get that there exists a natural number $m_1=m_1(\varepsilon)$ such that

$$\begin{aligned}
S_1\quad&=\quad\sum_{(k,j)\in I_2}\omega_{j,k}(x_0,y_0)\le\sum_{k=0}^{m+p}\sum_{j=0}^{m+p}\widetilde{p}_{m,k}(x_0)\widetilde{p}_{m,j}(y_0)\left|\varphi\left(\tfrac{k}{m},\tfrac{j}{m}\right)\right|\\
&<\quad\tfrac{\varepsilon}{4}\widetilde{B}_{m,m,p,q}(e_{00};x_0,y_0)<\tfrac{\varepsilon}{4}
\end{aligned}$$

for $m>m_1$.

Because $|\frac{j}{m}-y_o|\geq\delta$ implies $\delta^{-2}(\frac{j}{m}-y_0)^2\geq 1$, for the sum S_2, we get

$$\begin{aligned}
S_2 &= \sum_{(k,j)\in I_2}\omega_{j,k}(x_0,y_0)\\
&\leq \delta^{-2}\sum_{(k,j)\in I_2}(\tfrac{j}{m}-y_0)^2\widetilde{p}_{m,k}(x_0)\widetilde{p}_{m,j}(y_0)\left|\varphi(\tfrac{k}{m},\tfrac{j}{m})\right|\\
&\leq M\cdot\delta^{-2}\sum_{(k,j)\in I_2}(\tfrac{j}{m}-y_0)^2\widetilde{p}_{m,k}(x_0)\widetilde{p}_{m,j}(y_0)\leq\\
&\leq M\cdot\delta^{-2}\sum_{k=0}^{m+p}\sum_{j=0}^{m+p}(\tfrac{j}{m}-y_0)^2\widetilde{p}_{m,k}(x_0)\widetilde{p}_{m,j}(y_0)\\
&= M\cdot\delta^{-2}\widetilde{B}_{m,m,p,q}((\tau-y_0)^2;x_0,y_0),
\end{aligned}$$

where $M=\max\left\{|\varphi(x,y)|:(x,y)\in[0,1+p]\times[0,1+q]\right\}$.

Taking into account Lemma 3, we get that there exists a positive constant $M_2(m,q,y_0)$ such that

$$S_2\leq M_2(m,q,y_0)\,y_0\,(1-y_0)\cdot m^{-1}\ \delta^{-2}.$$

In the same way, applying Lemma 3 and Lemma 4, we obtain

$$S_3\leq M_3(m,q,x_0)\,x_0\,(1-x_0)\cdot m^{-1}\ \delta^{-2}$$

and

$$S_4\leq M_4(m,p,q,x_0,y_0)\,x_0\,y_0\,(1-x_0)(1-y_0)\cdot m^{-2}\ \delta^{-4}.$$

It follows that for given positive numbers ε, δ, M_i $(i=1,2,3)$ and $(x_0,y_0)\in[0,1+p]\times[0,1+q]$ there exists the natural numbers $m_2,\ m_3,\ m_4$ such that

$$\begin{aligned}
&S_2\leq M_2(m,q,y_0)\,y_0\,(1-y_0)\cdot m^{-1}\ \delta^{-2}<\tfrac{\varepsilon}{4},\quad \text{for } m>m_2,\\
&S_3\leq M_3(m,q,x_0)\,x_0\,(1-x_0)\cdot m^{-1}\ \delta^{-2}<\tfrac{\varepsilon}{4},\quad \text{for } m>m_3,\\
&S_4\leq M_4(m,p,q,x_0,y_0)\,x_0\,y_0\,(1-x_0)(1-y_0)\cdot m^{-2}\ \delta^{-4}<\tfrac{\varepsilon}{4},\quad \text{for } m>m_4.
\end{aligned}$$

Therefore, there exists the natural number $N=\max\{m_1,m_2,m_3,m_4\}$ such that for all natural numbers $n>N$ we have $S_1+S_2+S_3+S_4<\varepsilon$.

We can conclude that $\lim\limits_{m\to\infty}\widetilde{B}_{m,m,p,q}(\varphi;x_0,y_0)=0$, and the proof ends. $\square$

3. THE MAIN RESULT

Using the auxiliary results from section 2, we can now prove the Voronovskaja type theorem for the bivariate operators of Bernstein–Schurer.

Let us to denote by $C^{2,2}([0,1+p]\times[0,1+q])$ the space of all functions $f\in C([0,1+p]\times[0,1+q])$ with partial derivatives of first and second order belonging to the space $C([0,1+p]\times[0,1+q])$.

THEOREM 6. *Let* $f \in C^{2,2}([0,1+p]\times[0,1+q])$. *Then, for all* $(x,y) \in [0,1+p]\times[0,1+q]$ *we have*

$$\lim_{m\to\infty} m\left\{\widetilde{B}_{m,m,p,q}(f;x,y) - f(x,y)\right\} =$$

$$= p\ f'_x(x,y) + q\ f'_y(x,y) + +\tfrac{1}{2}\{x(1-x)f''_{x^2}(x,y) + y(1-y)f''_{y^2}(x,y)\}. \tag{13}$$

Proof. Let $(x_0,y_0) \in [0,1+p]\times[0,1+q]$ be a fixed point. By making use of the Taylor formula for $(t,\tau) \in [0,1+p]\times[0,1+q]$ with the Peano's form of the remainder term, we have

$$\begin{aligned} f(t,\tau) =& f(x_0,y_0) + (t-x)\cdot f'_x(x_0,y_0) + (\tau-y)\cdot f'_y(x_0,y_0) \\ &+ \tfrac{1}{2}\{(t-x_0)^2 f''_{x^2}(x_0,y_0) + 2(t-x_0)(\tau-y_0)f''_{xy}(x_0,y_0) + \\ &+ (\tau-y_0)^2 f''_{y^2}(x_0,y_0)\} + \varphi(t,\tau;x_0,y_0)\sqrt{(t-x_0)^4+(\tau-y_0)^4}, \end{aligned}$$

where $\varphi(\circ,*;x_0,y_0) \in C([0,1+p]\times[0,1+q])$ and $\lim\limits_{t\to x_0,\tau\to y_0}\varphi(t,\tau;x_0,y_0) = 0$.

Taking into account Lemma 3 we get

$$\begin{aligned} &\widetilde{B}_{m,m,p,q}(f(t,\tau);x_0,y_0) - f(x_0,y_0) = \\ &= p\cdot m^{-1}xf'_x(x_0,y_0) + qm^{-1}yf'_y(x_0,y_0) \\ &\quad + \tfrac{1}{2}\Big\{(p^2m^{-2}x_0^2 + (m+p)m^{-2}x_0(1-x_0))f''_{x_0^2}(x_0,y_0) \\ &\qquad + 2pqm^{-2}x_0y_0f''_{xy}(x_0,y_0) + q^2m^{-2}y_0^2 \\ &\qquad + (m+q)m^{-2}y_0(1-y_0))f''_{y_0^2}(x_0,y_0)\Big\} \\ &\quad + \widetilde{B}_{m,m,p,q}\left(\varphi(t,\tau;\ x_0,y_0)\sqrt{(t-x_0)^4+(\tau-y_0)^4};x_0,y\right). \end{aligned} \tag{14}$$

By Cauchy's inequality, it follows

$$\left|\widetilde{B}_{m,m,p,q}\left(\varphi(t,\tau;\ x_0,y_0)\sqrt{(t-x_0)^4+(\tau-y_0)^4};x_0,y_0\right)\right| \le$$

$$\le \left\{\widetilde{B}_{m,m,p,q}(\varphi^2(t,\tau;\ x_0,y_0))\right\}^{\frac{1}{2}}\left\{\widetilde{B}_{m,m,p,q}\left((t-x_0)^4+(\tau-y_0)^4;x_0,y_0\right)\right\}^{\frac{1}{2}}.$$

From Lemma 5 it follows

$$\lim_{m\to\infty}\widetilde{B}_{m,m,p,q}(\varphi^2(t,\tau;\ x_0,y_0);x_0,y_0) = 0 \tag{15}$$

and from Lemma 4 we get

$$\begin{aligned} &\widetilde{B}_{m,m,p,q}\left((t-x_0)^4+(\tau-y_0)^4;x_0,y_0\right) = \\ &= \widetilde{B}_{m,p}((t-x_0)^4;x_0) + \widetilde{B}_{m,q}((\tau-y_0)^4;y_0) \\ &\le \{M_1(x_0,p)+M_2(y_0,q)\}m^{-2}, \end{aligned} \tag{16}$$

where $M_1(x_0,p)$ and $M_2(y_0,q)$ are positive constants. Taking into account (15) and (16), by (14) we arrive to the desired result. □

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