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NEW CHEBYSHEV TYPE INEQUALITIES FOR SEQUENCES OF REAL NUMBERS

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Abstract. Some new inequalities of Chebyshev type for sequences of real numbers are pointed out.

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1. INTRODUCTION

For $\mathbf{p} = (p_1, ..., p_n), \mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$ n-tuples of real numbers, consider the Chebyshev functional

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) := P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i b_i, \tag{1}$$

where $P_n := \sum_{i=1}^{n} p_i$.

In 1882–1883, Chebyshev [1] and [2] proved that, if $\mathbf{a}$ and $\mathbf{b}$ are monotonic in the same (opposite) sense and $\mathbf{p}$ is nonnegative, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq (\leq)\, 0. \tag{2}$$

The inequality (2) was mentioned by Hardy, Littlewood and Polya in their book [3] in 1934 in the more general setting of synchronous sequences, i.e., if $\mathbf{a}, \mathbf{b}$ are synchronous (asynchronous), this means that

$$(a_i - a_j)(b_i - b_j) \geq (\leq)\, 0, \quad \text{for each } i, j \in \{1, ..., n\}, \tag{3}$$

then (2) holds true.

A relaxation of the synchronicity condition was provided by M. Biernacki in 1951, [4] who showed that for a nonnegative $\mathbf{p}$, if $\mathbf{a}, \mathbf{b}$ are monotonic in mean in the same sense, i.e. for $P_k := \sum_{i=1}^{k} p_i$,

$$\begin{aligned} &\tfrac{1}{P_k} \sum_{i=1}^{k} p_i a_i \leq (\geq)\, \tfrac{1}{P_{k+1}} \sum_{i=1}^{k+1} p_i a_i, \quad k \in \{1, ..., n-1\} \text{ and} \\ &\tfrac{1}{P_k} \sum_{i=1}^{k} p_i b_i \leq (\geq)\, \tfrac{1}{P_{k+1}} \sum_{i=1}^{k+1} p_i b_i, \quad k \in \{1, ..., n-1\}, \end{aligned} \tag{4}$$

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then (2) holds true for the " $\geq$ " sign. If they are monotonic in mean in the opposite sense, then (2) holds true for the " $\leq$ " sign.

For general real weights $\mathbf{p}$, Mitrinović and Pečarić has shown in [9] that the inequality (2) holds true if

$$0 \leq P_k \leq P_n, \quad \text{for } k \in \{1, ..., n-1\}, \tag{5}$$

and $\mathbf{a}, \mathbf{b}$ are monotonic in the same (opposite) sense.

The following identity is well known in the literature as Sonin's identity (see [5] and [6, p. 246])

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} p_i \left(P_n a_i - \sum_{j=1}^{n} p_j a_j \right) (b_i - b), \tag{6}$$

for any real number b.

Another well known identity in terms of double sums is the following one known in the literature as Korkine's identity (see [7] and [6, p. 242])

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j (a_i - a_j)(b_i - b_j). \tag{7}$$

The purpose of this work is to point out other identities of interest in obtaining Chebyshev's type inequalities. Some natural applications for studying the positivity of the Chebyshev's functional are mentioned.

For other results related to Chebyshev's functional see [10]–[12].

A survey on discrete Chebyshev's type inequalities is under preparation, [8].

2. THE IDENTITIES

The first result is embodied in the following

THEOREM 1. *Let $\mathbf{p} = (p_1, ..., p_n), \mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$ be n-tuples of real numbers. If we define*

$$P_i := \sum_{k=1}^{i} p_k, \bar{P}_i := P_n - P_i, \quad i \in \{1, ..., n-1\},$$

$$A_i(\mathbf{p}) := \sum_{k=1}^{i} p_k a_k, \bar{A}_i(\mathbf{p}) := A_n(\mathbf{p}) - A_i(\mathbf{p}), \quad i \in \{1, ..., n-1\},$$

then we have the identity

$$\begin{aligned} T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) &= \sum_{i=1}^{n-1} \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \cdot \Delta b_i \\ &= P_n \sum_{i=1}^{n-1} P_i \left(\frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right) \cdot \Delta b_i \; (\textit{if } P_n, P_i \neq 0, \quad i \in \{1, ..., n-1\}) \end{aligned} \tag{8}$$

$$= \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} - \frac{A_i(\mathbf{p})}{P_i} \right) \cdot \Delta b_i, \quad \left(if\ P_i, \bar{P}_i \neq 0, i \in \{1, ..., n-1\} \right),$$

where $\Delta b_i := b_{i+1} - b_i \, (i \in \{1, ..., n-1\})$ *is the forward difference.*

Proof. We use the following well known summation by parts formula

$$\sum_{l=p}^{q-1} d_l \Delta v_l = d_l v_l|_p^q - \sum_{l=p}^{q-1} v_{l+1} \Delta d_l, \tag{9}$$

where d_l, v_l are real numbers, $l = p, ..., q$ $(q > p; p, q$ are natural numbers$)$.

If we choose in (9), $p = 1, q = n, d_i = P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})$ and $v_i = b_i \, (i \in \{1, ..., n-1\})$, then we get

$$\begin{aligned}
&\sum_{i=1}^{n-1} (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \cdot \Delta b_i = \\
&= [P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})] \cdot b_i|_1^n - \sum_{i=1}^{n-1} \Delta (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \cdot b_{i+1} \\
&= [P_n A_n(\mathbf{p}) - P_n A_n(\mathbf{p})] \cdot b_n - [P_1 A_n(\mathbf{p}) - P_n A_1(\mathbf{p})] \cdot b_1 \\
&\quad - \sum_{i=1}^{n-1} [P_{i+1} A_n(\mathbf{p}) - P_n A_{i+1}(\mathbf{p}) - P_i A_n(\mathbf{p}) + P_n A_i(\mathbf{p})] \cdot b_{i+1} \\
&= P_n p_1 a_1 b_1 - p_1 b_1 A_n(\mathbf{p}) - \sum_{i=1}^{n-1} (p_{i+1} A_n(\mathbf{p}) - P_n p_{i+1} a_{i+1}) \cdot b_{i+1} \\
&= P_n p_1 a_1 b_1 - p_1 b_1 A_n(\mathbf{p}) - A_n(\mathbf{p}) \sum_{i=1}^{n-1} p_{i+1} b_{i+1} + P_n \sum_{i=1}^{n-1} p_{i+1} a_{i+1} b_{i+1} \\
&= P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i b_i \\
&= T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}),
\end{aligned}$$

which produce the first identity in (8).

The second and the third identity are obvious and we omit the details. □

Before we prove the second result, we need the following lemma providing an identity that is interesting in itself as well.

LEMMA 2. *Let* $\mathbf{p} = (p_1, ..., p_n)$ *and* $\mathbf{a} = (a_1, ..., a_n)$ *be n-tuples of real numbers. Then we have the equality*

$$\det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} = \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j, \tag{10}$$

for each $i \in \{1, ..., n-1\}$.

Proof. Define, for $i \in \{1, ..., n-1\}$,

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j.$$

We have

$$\begin{aligned} K(i) &= \sum_{j=1}^{i} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j \\ &= \sum_{j=1}^{i} P_j \bar{P}_i \cdot \Delta a_j + \sum_{j=i+1}^{n-1} P_i \bar{P}_j \cdot \Delta a_j \\ &= \bar{P}_i \sum_{j=1}^{i} P_j \cdot \Delta a_j + P_i \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j. \end{aligned} \tag{11}$$

Using the summation by parts formula, we have

$$\begin{aligned} \sum_{j=1}^{i} P_j \cdot \Delta a_j &= P_j \cdot a_j \big|_1^{i+1} - \sum_{j=1}^{i} (P_{j+1} - P_j) \cdot a_{j+1} \\ &= P_{i+1} a_{i+1} - p_1 a_1 - \sum_{j=1}^{i} p_{j+1} \cdot a_{j+1} \\ &= P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j \cdot a_j \end{aligned} \tag{12}$$

and

$$\begin{aligned} \sum_{j=i+1}^{n-1} \bar{P}_j \cdot \Delta a_j &= \bar{P}_j \cdot a_j \Big|_{i+1}^{n} - \sum_{j=i+1}^{n-1} \left(\bar{P}_{j+1} - \bar{P}_j \right) \cdot a_{j+1} \\ &= \bar{P}_n a_n - \bar{P}_{i+1} a_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j) \cdot a_{j+1} \\ &= -\bar{P}_{i+1} a_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1}. \end{aligned} \tag{13}$$

Using (12) and (13) we have

$$\begin{aligned} K(i) &= \bar{P}_i \Bigg(P_{i+1} a_{i+1} - \sum_{j=1}^{i+1} p_j \cdot a_j \Bigg) + P_i \Bigg(\sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_{i+1} a_{i+1} \Bigg) \\ &= \bar{P}_i P_{i+1} a_{i+1} - P_i \bar{P}_{i+1} a_{i+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} \end{aligned}$$

$$
\begin{aligned}
&= a_{i+1}\left(\left(P_n - P_i\right) P_{i+1} - P_i\left(P_n - P_{i+1}\right)\right) \\
&\quad + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j \\
&= P_n p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j \\
&= \left(P_i + \bar{P}_i\right) p_{i+1} a_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1} \cdot a_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j \cdot a_j \\
&= P_i \sum_{j=i+1}^{n-1} p_j \cdot a_j - \bar{P}_i \sum_{j=1}^{i} p_j \cdot a_j \\
&= P_i \bar{A}_i(\mathbf{p}) - \bar{P}_i A_i(\mathbf{p}) \\
&= \det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix}
\end{aligned}
$$

and the identity is proved. □

We are able now to state and prove the second identity for the Chebyshev functional

Theorem 3. *With the assumptions of Theorem* 1, *we have the equality*

$$
T_n(\mathbf{p};\mathbf{a},\mathbf{b}) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \cdot \Delta a_j \cdot \Delta b_j. \tag{14}
$$

The proof is obvious by Theorem 1 and Lemma 2.

Remark 1. The identity (14) was stated without a proof in paper [9]. It also may be found in [6, p. 281], again without a proof. □

3. SOME NEW INEQUALITIES

We may point out the following result concerning the positivity of the Chebyshev functional

Theorem 4. *Let* $\mathbf{p} = (p_1, ..., p_n), \mathbf{a} = (a_1, ..., a_n)$ *and* $\mathbf{b} = (b_1, ..., b_n)$ *be* n*-tuples of real numbers. If* $\mathbf{b}$ *is monotonic nondecreasing and either*

(i) $\det \begin{pmatrix} P_i & P_n \\ A_i(\mathbf{p}) & A_n(\mathbf{p}) \end{pmatrix} \geq 0$ *for each* $i \in \{1, ..., n-1\}$;

or

(ii) $P_i > 0$ *for any* $i \in \{1, ..., n\}$ *and*

$$\frac{A_n(\mathbf{p})}{P_n} \geq \frac{A_i(\mathbf{p})}{P_i} \text{ for each } i \in \{1, ..., n-1\};$$

or

(iii) $0 < P_i < P_n$ *for every* $i \in \{1, ..., n-1\}$ *and*

$$\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} \geq \frac{A_i(\mathbf{p})}{P_i} \text{ for each } i \in \{1, ..., n-1\};$$

then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq 0. \tag{15}$$

If $\mathbf{b}$ *is monotonic nonincreasing and either* (i) *or* (ii) *or* (iii) *from above holds, then the reverse inequality in* (15) *holds true.*

The proof of the theorem follows from the identities incorporated in Theorem 1 and we omit the details.

Using the second theorem, we may state the following result as well

THEOREM 5. *Let* $\mathbf{p} = (p_1, ..., p_n)$, $\mathbf{a} = (a_1, ..., a_n)$ *and* $\mathbf{b} = (b_1, ..., b_n)$ *be* n*-tuples of real numbers. If* $\mathbf{a}$ *and* $\mathbf{b}$ *are monotonic in the same sense and*

$$P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}} \geq 0 \text{ for each } i, j \in \{1, ..., n-1\}, \tag{16}$$

then (15) *holds true. If* $\mathbf{a}$ *and* $\mathbf{b}$ *are monotonic in the opposite sense and the condition* (16) *is valid, then the reverse inequality in* (15) *holds true.*

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