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ON THE MODIFIED BETA APPROXIMATING OPERATORS OF FIRST KIND

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Abstract. We define a general linear operator from which we obtain as special case the modified beta first kind operator

$$(B_{p,q}f)(x) = \tfrac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f\left(\tfrac{B(p,q)}{B(p+a,q)} t^a x\right) \mathrm{d}t.$$

We consider here only the cases $a = 1$ and $a = -1$.

We obtain several positive linear operators as particular cases of this modified beta first kind operator.

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1. INTRODUCTION

Many authors introduced and studied positive linear operators, using Euler's beta function of first kind: [1]–[4].

Euler's beta function of first kind is defined for $p > 0$, $q > 0$, by the following formula

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t \tag{1}$$

The beta transform of the function f is defined by the following formula

$$\mathcal{B}_{p,q} f = \tfrac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t)\mathrm{d}t.$$

The modified beta operator is defined for $x \geq 0$ by the following formula

$$(\mathcal{B}_{p,q}f)(x) = \tfrac{1}{B(p,q)} \int_0^1 t^{p-q}(1-t)^{q-1} f(tx)\mathrm{d}t.$$

We shall define a more general linear operator from which we obtain as a particular case the modified beta first kind operator.

For $a, b \in \mathbb{R}$ and $x \geq 0$, we define the (a,b)-modified beta operator

$$(\mathcal{B}^{(a,b)}_{p,q} f)(x) = \tfrac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f\big(\tfrac{B(p,q)}{B(p+a,q+b)} t^a (1-t)^b x\big)\mathrm{d}t, \tag{2}$$

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where $B(\cdot,\cdot)$ is the beta function (1) and f is any real measurable function defined on $(0,\infty)$ such that

$$(\mathcal{B}_{p,q}^{(a,b)}|f|)(x) < \infty.$$

2. THE MODIFIED BETA FIRST KIND OPERATORS

If we put in (2) $b = 0$ we obtain the modified beta first kind operator

$$(\mathcal{B}_{p,q}^{(a)}f)(x) = \tfrac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f\big(\tfrac{B(p,q)}{B(p+a,q)}t^a x\big)\mathrm{d}t, \tag{3}$$

where $B(\cdot,\cdot)$ is the beta function (1) and f is any real measurable function defined on $[0,\infty)$ such that $(\mathcal{B}_{p,q}^{(a)}|f|)(x) < \infty$.

One observe that $\mathcal{B}_{p,q}^{(a)}$ is a positive linear operator and

$$(\mathcal{B}_{p,q}^{(a)}e_1)(x) = x.$$

2.1. Case $a = 1$. If we choose in (3) $a = 1$ we obtain the modified beta first kind operator

$$(\mathcal{B}_{p,q}f)(x) = (\mathcal{B}_{p,q}^{(1)}f)(x) = \tfrac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f\big(\tfrac{p+q}{p}\cdot tx\big)\mathrm{d}t. \tag{4}$$

REMARK 1. If we choose in (4) $p > 0$ and $q > 0$ such that $\frac{p}{p+q} = x$, $x \in (0,1)$, then we obtain the operator (2.5) considered by the author in [4]. □

LEMMA 1. *The moments of order k of the operator $\mathcal{B}_{p,q}$ have the following values*

$$(\mathcal{B}_{p,q}e_k)(x) = \big(\tfrac{p+q}{p}\big)^k \tfrac{(p)_k}{(p+q)_k}x^k.$$

Proof.

$$\begin{aligned}(\mathcal{B}_{p,q}e_k)(x) &= \tfrac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}\big(\tfrac{p+q}{p}\big)^k t^k x^k \mathrm{d}t\\ &= \big(\tfrac{p+q}{p}\big)^k \tfrac{x^k}{B(p,q)}\int_0^1 t^{p+k-1}(1-t)^{q-1}\mathrm{d}t\\ &= \big(\tfrac{p+q}{p}\big)^k \tfrac{B(p+k,q)}{B(p,q)}x^k\\ &= \big(\tfrac{p+q}{p}\big)^k x^k \tfrac{\Gamma(p+k)\Gamma(q)}{\Gamma(p+q+k)}\tfrac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\\ &= \big(\tfrac{p+q}{p}\big)^k \tfrac{(p)_k}{(p+q)_k}x^k.\end{aligned}$$

For $k = 1$, $k = 2$ we obtain

$$\begin{aligned}
(\mathcal{B}_{p,q}e_1)(x) =& x,\\
(\mathcal{B}_{p,q}e_2)(x) =& \left(\tfrac{p+q}{p}\right)^2 \tfrac{p(p+1)}{(p+q)(p+q+1)} x^2 = \tfrac{(p+q)(p+1)}{p(p+q+1)} x^2,\\
\mathcal{B}_{p,q}((t-x)^2;x) =& \left(\tfrac{(p+q)(p+1)}{p(p+q+1)} - 1\right) x^2\\
=& \tfrac{p^2+p+pq+q-p^2-pq-p}{p(p+q+1)} x^2\\
=& \tfrac{q}{p(p+q+1)} x^2.
\end{aligned}$$

□

Consequently, we obtain

$$\begin{aligned}
(\mathcal{B}_{p,q}e_1)(x) =& x,\\
(\mathcal{B}_{p,q}e_2)(x) =& \tfrac{p+1}{p} \cdot \tfrac{p+q}{p+q+1} x^2,\\
\mathcal{B}_{p,q}((t-x)^2;x) =& \tfrac{qx^2}{p(p+q+1)}.
\end{aligned}$$

Special cases

A) If we put in (4) $p = n-1$ and $q = \alpha$, $\alpha > 0$ we obtain the positive linear operator

$$(\mathcal{B}_n^{(\alpha)} f)(x) = \tfrac{1}{B(n,\alpha)} \int_0^1 t^{n-1}(1-t)^{\alpha-1} f\left(\tfrac{n+\alpha-1}{n-1} tx\right) \mathrm{d}t. \tag{5}$$

COROLLARY 2. *The following relation holds:*

$$\mathcal{B}_n^{(\alpha)}((t-x)^2;x) = \tfrac{\alpha x^2}{(n-1)(n+\alpha)}.$$

Proof. It is obtained from Lemma 1 for $p = n-1$ and $q = \alpha$, $\alpha > 0$. □

REMARK 2. For $\alpha = 1$ we obtain

$$\mathcal{B}_n((t-x)^2;x) = \tfrac{x^2}{n^2-1}.$$

□

B) Another operator it is obtained by (4) for $p = nx$, $n \in \mathbb{N}$, $q = \alpha$, $\alpha > 0$:

$$(\overline{\mathcal{B}}_n^{(\alpha)} f)(x) = \tfrac{1}{B(nx,\alpha)} \int_0^1 t^{nx-1}(1-t)^{\alpha-1} f\left(\tfrac{nx+\alpha}{n} t\right) \mathrm{d}t. \tag{6}$$

COROLLARY 3.

$$\overline{\mathcal{B}}_n^{(\alpha)}((t-x)^2;x) = \tfrac{\alpha x}{n(nx+\alpha+1)}.$$

Proof. It is obtained from Lemma 1 for $p = nx$ and $q = \alpha$, $\alpha > 0$. □

2.2. Case $a = -1$. If we put $a = -1$ in (4) we obtain the modified beta first kind operator

$$(\mathbf{B}_{p,q} f)(x) = (\mathcal{B}_{p,q}^{(-1)} f)(x) = \tfrac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f\left(\tfrac{p-1}{p+q-1} \cdot \tfrac{x}{t}\right) \mathrm{d}t. \tag{7}$$

REMARK 3. If we choose in (7) $p > 0$ and $q > 0$ such that $\frac{p+q-1}{p-1} = x$, $x > 1$ then we obtain the operator (4.5) considered by the author in [4]. □

Lemma 4. *The moments of order k $(1\le k<p)$ of the operator $\boldsymbol{B}_{p,q}$ have the following values*

$$(\boldsymbol{B}_{p,q}e_k)(x)=\tfrac{(p+q-1)\dots(p+q-k)}{(p-1)\dots(p-k)}\left(\tfrac{p-1}{p+q-1}\right)^k x^k,\quad 1\le k<p.$$

Proof.

$$\begin{aligned}(\mathbf{B}_{p,q}e_k)(x)=&\tfrac{1}{B(p,q)}\int_0^1 t^{p-q}(1-t)^{q-1}\big(\tfrac{p-1}{p+q-1}\big)^k\tfrac{x^k}{t^k}\mathrm{d}t\\ =&\big(\tfrac{p-1}{p+q-1}\big)^k\tfrac{x^k}{B(p,q)}\int_0^1 t^{p-k-1}(1-t)^{q-1}\mathrm{d}t\\ =&\big(\tfrac{p-1}{p+q-1}\big)^k\tfrac{B(p-k,q)}{B(p,q)}x^k\\ =&\big(\tfrac{p-1}{p+q-1}\big)^k\tfrac{\Gamma(p-k)\Gamma(q)}{\Gamma(p+q-k)}\cdot\tfrac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^k\\ =&\big(\tfrac{p-1}{p+q-1}\big)^k\tfrac{(p+q-1)\dots(p+q-k)}{(p-1)\dots(p-k)}x^k.\end{aligned}$$

□

Consequently, we obtain:

$$\begin{aligned}(\mathbf{B}_{p,q}e_1)(x)=&x,\\ (\mathbf{B}_{p,q}e_2)(x)=&\tfrac{p-1}{p-2}\cdot\tfrac{p+q-2}{p+q-1}x^2,\quad p>2,\\ \mathbf{B}_{p,q}((t-x)^2;x)=&\tfrac{q}{(p-2)(p+q-1)}x^2,\quad p>2.\end{aligned}$$

Special cases

A) If we put in (7) $p=n+1$, $q=\alpha$, $\alpha>0$ we obtain the positive linear operator

$$(\mathbf{B}_n^{(\alpha)}f)(x)=\tfrac{1}{B(n+1,\alpha)}\int_0^1 t^n(1-t)^{\alpha-1}f\big(\tfrac{n}{n+\alpha}\cdot\tfrac{x}{t}\big)\mathrm{d}t.$$

Corollary 5.

$$\boldsymbol{B}_n^{(\alpha)}((t-x)^2;x)=\tfrac{\alpha x^2}{(n-1)(n+\alpha)}.$$

Proof. It is obtained from Lemma 4 for $p=n+1$, $q=\alpha$. □

Remark 4. For $\alpha=1$ we obtain

$$\mathbf{B}_n((t-x)^2;x)=\tfrac{x^2}{n^2-1}.$$

□

B) Another operator it is obtained by (7) for $p=nx+2$, $n\in\mathbb{N}$, $q=\alpha$, $\alpha>0$.

$$(\overline{\mathbf{B}}_n^{(\alpha)}f)(x)=\tfrac{1}{B(nx+2,\alpha)}\int_0^1 t^{nx+1}(1-t)^{\alpha-1}f\big(\tfrac{nx+1}{nx+\alpha+1}\cdot\tfrac{x}{t}\big)\mathrm{d}t.$$

Corollary 6. *One has*

$$\overline{\mathbf{B}}_n^{(\alpha)}((t-x)^2;x)=\tfrac{\alpha x^2}{nx(nx+\alpha+1)}.$$

Proof. It is obtained from Lemma 4 for $p=nx+2$, $q=\alpha$. □

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