COMBINED SHEPARD OPERATORS
WITH CHEBYSHEV NODES

CRISTINA O. OȘAN∗ and RADU T. TRÎMBIȚAŞ†

Abstract. In this paper we study combined Shepard-Lagrange univariate interpolation operator

\[ S_{L,m}^{n} (Y; f, x) := \sum_{k=0}^{n+1} |x - y_{n,k}|^{-\mu} \frac{(L_m f)(x, y_{n,k})}{\sum_{k=0}^{n+1} |x - y_{n,k}|^{-\mu}}, \]

where \((y_{n,k})\) are the interpolation nodes and \((L_m f)(x; y_{n,k})\) is the Lagrange interpolation polynomial with nodes \(y_{n,k}, y_{n,k+1}, y_{n,k+2}, \ldots, y_{n,k+m}\), when the interpolation nodes \((y_{n,k})_{k=1}^{n}\) are the zeros of first kind Chebyshev polynomial completed with \(y_{n,0} = -1\) and \(y_{n,n+1} = 1\). We give a direct proof for error estimation and some numerical examples.

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1. INTRODUCTION

Let \( Y = \{y_{n,i} \in [-1, 1]; i = 0, n + 1; n \in \mathbb{N}\} \) be an infinite matrix where each row is a set of distinct points in \([-1, 1]\). For \( f \in C^m([-1, 1]) \) the Shepard-Lagrange operator is defined by

\[ S_{L,m}^{n} (Y; f, x) := S_{L,m}^{n} (f, x) = \sum_{k=0}^{n+1} |x - y_{n,k}|^{-\mu} \frac{(L_m f)(x, y_{n,k})}{\sum_{k=0}^{n+1} |x - y_{n,k}|^{-\mu}}, \]

where \(m \in \mathbb{N}, m < n\) is prescribed.

The Shepard-Lagrange operator was treated in \([1]\) and \([6]\). Its most important properties are: it preserves polynomials of degree \(m\), i.e.,

\[ S_{L,m}^{n} (e_j; x) = e_j(x), \quad j = 0, m, \]

where \(e_j(x) = x^j\). Also,

\[ S_{L,m}^{n} (f; y_{n,k}) = f(y_{n,k}), \quad k = 1, n. \]

∗"Babeș-Bolyai" University, Faculty of Mathematics and Computer Science, str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania, e-mail: cristina@math.ubbcluj.ro.
†"Babeș-Bolyai" University, Faculty of Mathematics and Computer Science, str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania, e-mail: tradu@math.ubbcluj.ro.
In this paper we give an error estimation analogous to that given in [3], but the present proof is direct and exploits the properties of Cebyshev nodes and Lagrange interpolation polynomials with such nodes.

2. ERROR ESTIMATION

If \( f \in C^p([a, b]), p \in \mathbb{N}, p < m \), and \((L_m f)\) is the \(m\)-th degree Lagrange interpolation polynomial with nodes \(x_0, x_1, \ldots, x_m \in [a, b] \), the following error estimation holds (see [3])

\[
\|f - L_m f\|_\infty \leq C_p (b - a)^p \frac{\log m}{m^p} \omega(f^{(p)}; \frac{b-a}{m-p}),
\]

where \( C_p > 0 \).

Let \((y_{n,k})_{k=0}^{d}\) be the set of the roots of first kind \(n\)-th degree Chebyshev polynomial, completed with \(y_{n,0} = -1\) and \(y_{n,n+1} = 1\). For \(k = 1, n, y_{n,k} = \cos \theta_{n,k}\), where \(\theta_{n,k} = \frac{(2k - 1)\pi}{2n}\).

Remark 1. We have

\[
|\theta_{n,k} - \theta_{n,k+1}| = \frac{\pi}{n}.
\]

The following theorem holds:

Theorem 1. If \( f \in C^p([-1, 1]) \) and \( \mu > p + 1 \), we have

\[
|f(x) - S_{L_m}^{L_m}(f; x)| \leq \frac{(1-x^2)^p}{n^{\mu-1}} \int_{n-1}^1 \omega(f^{(p)}; \frac{1-x^2}{t^{p-\mu}}) dt.
\]

Proof. Let \(y_{n,d}\) be the closest point to \(x\). We share the nod set \((y_{n,k})\) into three classes: \(k = 0, d - m; k = d - m + 1, d; k = d + 1, n + 1\).


\[
|f(x) - S_{L_m}^{L_m}(f; x)| \leq \sum_{k=0}^{n+1} \left| \frac{x - y_{n,d}}{x - y_{n,k}} \right|^\mu |f(x) - (L_m f)(x, y_{n,k})| \leq C_p \frac{\log m}{m^p} \sum_{k=0}^{d} \left| \frac{x - y_{n,d}}{x - y_{n,k}} \right|^\mu |x - y_{n,k}| \omega(f^{(p)}; \frac{|x - y_{n,k}|}{m - p}) + \sum_{k=d-m+1}^{d} \left| \frac{x - y_{n,d}}{x - y_{n,k}} \right|^\mu |y_{n,k+m} - y_{n,k}| \omega(f^{(p)}; \frac{|y_{n,k+m} - y_{n,k}|}{m - p}) + \sum_{k=d+1}^{n+1} \left| \frac{x - y_{n,d}}{x - y_{n,k}} \right|^\mu |x - y_{n,k+m}| \omega(f^{(p)}; \frac{|x - y_{n,k+m}|}{m - p})
\]

:= C_p \frac{\log m}{m^p} (S_1 + S_2 + S_3).
We also have

\[ |x - y_{n,d}| \leq \frac{\pi}{n} \sqrt{1 - x^2}, \]

(4)

\[ |x - y_{n,k}| \geq \frac{\pi}{n} |k - d| \sqrt{1 - x^2}, \]

(5)

\[ |x - y_{n,k+m}| \leq |x - y_{n,k}| + \frac{\pi m}{n}. \]

(6)

We shall also use the following properties for modulus of continuity

\[ |x - y| \leq \frac{\pi}{n} \sqrt{1 - x^2} \]

\[ |x - y| \leq \frac{\pi}{n} \sqrt{1 - x^2} \]

\[ \delta_2 \geq \delta_1 \Rightarrow \frac{\omega(f \delta_2)}{\delta_2} \leq \frac{\omega(f \delta_1)}{\delta_1}, \]

(7)

\[ \omega(f; \lambda \delta) \leq (1 + \lambda)\omega(f; \delta) \text{ if } \lambda \in \mathbb{R}_+, \]

(8)

\[ \delta_1 \leq \delta_2 \Rightarrow \omega(f; \delta_1) \leq \omega(f; \delta_2), \]

(9)

\[ \omega(f; \delta_1 + \delta_2) \leq \omega(f; \delta_1) + \omega(f; \delta_2). \]

(10)

Now, it follows the estimations for \( S_1, S_2, \) and \( S_3. \)

\[ S_1 = \sum_{k=0}^{d-m} \left| \frac{x-y_{n,d}}{x-y_{n,k}} \right|^p \frac{1}{|k-d|m-p} \omega(f; \frac{|x-y_{n,k}|}{m-p}). \]

We obtain

\[ S_1 \leq c_1 \left( \frac{\sqrt{1 - x^2}}{n^2} \right)^p \cdot \sum_{k=0}^{d-m} \frac{1}{|k-d|m-p} \omega(f; \frac{|x-y_{n,k}|}{m-p}). \]

with

\[ c_1 = 2 \pi^p(\pi + 1) \left( 1 + \frac{1}{m-p} \right). \]

\[ S_2 = \sum_{k=d-m+1}^{d} \left| \frac{x-y_{n,d}}{x-y_{n,k}} \right|^p \frac{1}{|y_{n,k+m} - y_{n,k}|} \omega(f; \frac{|y_{n,k+m} - y_{n,k}|}{m-p}). \]

Since \( |y_{n,k+m} - y_{n,k}| \leq \frac{m \pi}{n} \sqrt{1 - x^2} \) we have

\[ \omega(f; \frac{|y_{n,k+m} - y_{n,k}|}{m-p}) \leq (1 + \frac{1}{m-p})(m \pi + 1)\omega(f; \frac{\sqrt{1 - x^2}}{n}). \]

We obtain

\[ S_2 \leq c_2 \left( \frac{\sqrt{1 - x^2}}{n} \right)^p \omega(f; \frac{\sqrt{1 - x^2}}{n}) \]

with

\[ c_2 = (m \pi)^p(m \pi + 1)^p(m-1)(1 + \frac{1}{m-p}). \]

\[ S_3 = \sum_{k=d+1}^{n} \left| \frac{x-y_{n,d}}{x-y_{n,k}} \right|^p \frac{1}{|x-y_{n,k+m}|} \omega(f; \frac{|x-y_{n,k+m}|}{m-p}). \]
For $S_3$ we remark that

$$|x - y_{n,k+m}|^p \leq (|x - y_{n,k}| + |y_{n,k+m} - y_{n,k}|)^p = \sum_{r=0}^{p} \binom{p}{r} |x - y_{n,k}|^{p-r} |y_{n,k+m} - y_{n,k}|^r.$$ 

Now, we have

$$S_3 \leq |x - y_{n,d}|^{\mu} \sum_{k=d+1}^{n+1} \sum_{r=0}^{p} \binom{p}{r} \frac{|y_{n,k} - y_{n,k+m}|^r}{|x - y_{n,k}|^{\mu-p+r}} \frac{\omega(f(p)}{m-p} 
\leq \pi^p \left(\frac{\sqrt{1-x^2}}{n}\right)^p \left(\sum_{r=0}^{p} \binom{p}{r} m^r\right) \sum_{k=d+1}^{n+1} \sum_{r=0}^{p} \frac{1}{|k-d|^{\mu-p}} \frac{\omega(f(p}}{m-p} 
\leq (1 + \frac{1}{m-p}) \pi^p (m+1)^p \left(\frac{\sqrt{1-x^2}}{n}\right)^p \sum_{k=d+1}^{n+1} \sum_{r=0}^{p} \frac{1}{|k-d|^{\mu-p}} \frac{\omega(f(p)}{m-p} |x - y_{n,k+m}|.$$ 

We remark that for $\mu - p > 1$, $\sum_{k=d+1}^{n+1} |k - d|^{\mu-p}$ is bounded by $M$.

From this observation and (10), (7) we obtain

$$\omega(f(p); |x - y_{n,k+m}|) \leq M (m\pi + 1) \omega(f(p; \sqrt{1-x^2}) + 2(\pi + 1) \omega(f(p; |k-d|/n) \sqrt{1-x^2}).$$

In conclusion, we obtain for $S_3$ the following inequality

$$S_3 \leq \left(\frac{\sqrt{1-x^2}}{n}\right)^p \left[c_3 \omega(f(p; \sqrt{1-x^2}) + c_4 \sum_{k=d+1}^{n+1} \frac{\omega(f(p; |k-d|/n) \sqrt{1-x^2})}{|k-d|^{\mu-p}} \right]$$

with the constants

$$c_3 = \pi^p (m+1)^p M (m\pi + 1) (1 + \frac{1}{m-p}),
\quad c_4 = 2\pi^p (m+1)^p (\pi + 1) (1 + \frac{1}{m-p}).$$

Since

$$\omega(f(p; \sqrt{1-x^2}) \leq \frac{1}{n^{\mu-p}} \int_{n-1}^{1} \frac{\omega(f(p; t\sqrt{1-x^2})}{t^{\mu-p}} dt$$

and

$$S_1 + S_2 + S_3 \leq \left(\frac{\sqrt{1-x^2}}{n}\right)^p \left[C(c_2, c_3) \cdot \omega(f(p; \sqrt{1-x^2}) + C(c_1, c_4) \cdot \omega(f(p; |k-d|/n) \sqrt{1-x^2}) \right]$$

(3) follows. □
3. EXAMPLES AND GRAPHS

Let us consider the function $f : [-1, 1] \to \mathbb{R}$,

$$f(x) = \sin \pi x.$$ 

Its graph appears in figure 1, together with the Shepard-Lagrange approximation functions for $\mu \in \{2, 4\}$, $n = 16$ and $m \in \{1, 2\}$.

Fig. 1. Graph of $f$ and various Shepard-Lagrange interpolants.
REFERENCES


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