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COMBINED SHEPARD OPERATORS WITH CHEBYSHEV NODES

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Abstract. In this paper we study combined Shepard-Lagrange univariate interpolation operator

$$S_{n,\mu}^{L,m}(Y;f,x) := S_{n,\mu}^{L,m}(f,x) = \frac{\sum_{k=0}^{n+1} |x-y_{n,k}|^{-\mu}\,(L_m f)(x,y_{n,k})}{\sum_{k=0}^{n+1} |x-y_{n,k}|^{-\mu}},$$

where $(y_{n,k})$ are the interpolation nodes and $(L_m f)(x;y_{n,k})$ is the Lagrange interpolation polynomial with nodes $y_{n,k}$, $y_{n,k+1}$, $y_{n,k+2}, \ldots$, $y_{n,k+m}$, when the interpolation nodes $(y_{n,k})_{k=\overline{1,n}}$ are the zeros of first kind Chebyshev polynomial completed with $y_{n,0} = -1$ and $y_{n,n+1} = 1$. We give a direct proof for error estimation and some numerical examples.

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1. INTRODUCTION

Let $Y = \{y_{n,i} \in [-1,1];\ i = \overline{0,n+1};\ n \in \mathbb{N}\}$ be an infinite matrix where each row is a set of distinct points in $[-1,1]$. For $f \in C^m([-1,1])$ the Shepard-Lagrange operator is defined by

$$S_{n,\mu}^{L,m}(Y;f,x) := S_{n,\mu}^{L,m}(f,x) = \frac{\sum_{k=0}^{n+1} |x-y_{n,k}|^{-\mu}\,(L_m f)(x,y_{n,k})}{\sum_{k=0}^{n+1} |x-y_{n,k}|^{-\mu}}, \tag{1}$$

where $m \in \mathbb{N}$, $m < n$ is prescribed.

The Shepard-Lagrange operator was treated in [1] and [6]. Its most important properties are: it preserves polynomials of degree m, i.e.,

$$S_{n,\mu}^{L,m}(e_j;x) = e_j(x), \quad j = \overline{0,m},$$

where $e_j(x) = x^i$. Also,

$$S_{n,\mu}^{L,m}(f;y_{n,k}) = f(y_{n,k}), \quad k = \overline{1,n}.$$

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In this paper we give an error estimation analogous to that given in [6], but the present proof is direct and exploits the properties of Cebyshev nodes and Lagrange interpolation polynomials with such nodes.

2. ERROR ESTIMATION

If $f \in C^p([a,\, b])$, $p \in \mathbb{N}$, $p < m$, and $(L_m f)$ is the m-th degree Lagrange interpolation polynomial with nodes $x_0,\, x_1,\, \ldots,\, x_m \in [a,\, b]$, the following error estimation holds (see [3])

$$\|f - L_m f\|_\infty \le C_p (b-a)^p \ \tfrac{\log m}{m^p} \omega\big(f^{(p)}; \tfrac{b-a}{m-p}\big), \tag{2}$$

where $C_p > 0$.

Let $(y_{n,k})_{k=\overline{0,n+1}}$ be the set of the roots of first kind n-th degree Chebyshev polynomial, completed with $y_{n,0} = -1$ and $y_{n,n+1} = 1$. For $k = \overline{1,n}$, $y_{n,k} = \cos\theta_{n,k}$, where $\theta_{n,k} = (2k-1)\pi/(2n)$.

Remark 1. We have

$$|\theta_{n,k} - \theta_{n,k+1}| = \tfrac{\pi}{n}. \qquad \square$$

The following theorem holds:

Theorem 1. *If $f \in C^p([-1,1])$ and $\mu > p+1$, we have*

$$\Big| f(x) - S^{L,m}_{n,\mu}(f;x) \Big| \le \tfrac{(1-x^2)^p}{n^{\mu-1}} \int_{n^{-1}}^{1} \tfrac{\omega(f^{(p)}; t\sqrt{1-x^2})}{t^{\mu-p}} \mathrm{d}t. \tag{3}$$

Proof. Let $y_{n,d}$ be the closest point to x. We share the nod set $(y_{n,k})$ into three classes: $k = \overline{0, d-m}$; $k = \overline{d-m+1, d}$; $k = \overline{d+1, n+1}$.

Equations (1) and (2) imply

$$\begin{aligned}
&\Big| f(x) - S^{L,m}_{n,\mu}(f;x) \Big| \le \\
&\le \sum_{k=0}^{n+1} \Big| \tfrac{x-y_{n,d}}{x-y_{n,k}} \Big|^\mu |f(x) - (L_m f)(x, y_{n,k})| \\
&\le C_p \tfrac{\log m}{m^p} \Bigg[\sum_{k=0}^{d-m} \Big| \tfrac{x-y_{n,d}}{x-y_{n,k}} \Big|^\mu |x - y_{n,k}|^p\, \omega\big(f^{(p)}; \tfrac{|x-y_{n,k}|}{m-p}\big) \\
&\qquad + \sum_{k=d-m+1}^{d} \Big| \tfrac{x-y_{n,d}}{x-y_{n,k}} \Big|^\mu |y_{n,k+m} - y_{n,k}|^p\, \omega\big(f^{(p)}; \tfrac{|y_{n,k+m}-y_{n,k}|}{m-p}\big) + \\
&\qquad + \sum_{k=d+1}^{n+1} \Big| \tfrac{x-y_{n,d}}{x-y_{n,k}} \Big|^\mu |x - y_{n,k+m}|^p\, \omega\big(f^{(p)}; \tfrac{|x-y_{n,k+m}|}{m-p}\big) \Bigg] \\
&:= C_p \tfrac{\log m}{m^p} (S_1 + S_2 + S_3).
\end{aligned}$$

We also have

$$|x-y_{n,d}| \leq \tfrac{\pi}{n}\sqrt{1-x^2}, \tag{4}$$

$$|x-y_{n,k}| \geq \tfrac{\pi}{n}\,|k-d|\,\sqrt{1-x^2}, \tag{5}$$

$$|x-y_{n,k+m}| \leq |x-y_{n,k}| + \pi\tfrac{m}{n}. \tag{6}$$

We shall also use the following properties for modulus of continuity

$$\delta_2 \geq \delta_1 \Rightarrow \tfrac{\omega(f;\delta_2)}{\delta_2} \leq \tfrac{\omega(f;\delta_1)}{\delta_1}, \tag{7}$$

$$\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta) \text{ if } \lambda \in \mathbb{R}_+, \tag{8}$$

$$\delta_1 \leq \delta_2 \Rightarrow \omega(f;\delta_1) \leq \omega(f;\delta_2), \tag{9}$$

$$\omega(f;\delta_1+\delta_2) \leq \omega(f;\delta_1)+\omega(f;\delta_2). \tag{10}$$

Now, it follows the estimations for S_1, S_2, and S_3.

$$S_1 = \sum_{k=0}^{d-m} \left|\tfrac{x-y_{n,d}}{x-y_{n,k}}\right|^{\mu} |x-y_{n,k}|^p\, \omega\big(f^{(p)}; \tfrac{|x-y_{n,k}|}{m-p}\big).$$

From (7) for $\delta_1 := \frac{\pi}{n}\,|k-d|\,\sqrt{1-x^2}$ and $\delta_2 := |x-y_{n,k}|$ we have

$$\frac{\omega\big(f^{(p)};|x-y_{n,k}|\big)}{|x-y_{n,k}|} \leq \frac{2n(\pi+1)}{\pi|k-d|\sqrt{1-x^2}}.$$

We obtain

$$S_1 \leq c_1\big(\tfrac{\sqrt{1-x^2}}{n}\big)^p \cdot \sum_{k=0}^{d-m} \tfrac{1}{|k-d|^{\mu-p}}\ \omega\big(f^{(p)}; \tfrac{|k-d|}{n}\sqrt{1-x^2}\big)$$

with

$$c_1 = 2\pi^p(\pi+1)\left(1+\tfrac{1}{m-p}\right),$$

$$S_2 = \sum_{k=d-m+1}^{d} \left|\tfrac{x-y_{n,d}}{x-y_{n,k}}\right|^{\mu} |y_{n,k+m}-y_{n,k}|^p\, \omega\big(f^{(p)}; \tfrac{|y_{n,k+m}-y_{n,k}|}{m-p}\big).$$

Since $|y_{n,k+m}-y_{n,k}| \leq \frac{m\pi}{n}\sqrt{1-x^2}$ we have

$$\omega\big(f^{(p)}; \tfrac{|y_{n,k+m}-y_{n,k}|}{m-p}\big) \leq \big(1+\tfrac{1}{m-p}\big)(m\pi+1)\omega\big(f^{(p)}; \tfrac{\sqrt{1-x^2}}{n}\big).$$

We obtain

$$S_2 \leq c_2\big(\tfrac{\sqrt{1-x^2}}{n}\big)^p \omega\big(f^{(p)}; \tfrac{\sqrt{1-x^2}}{n}\big)$$

with

$$c_2 = (m\pi)^p(m\pi+1)^p(m-1)\big(1+\tfrac{1}{m-p}\big),$$

$$S_3 = \sum_{k=d+1}^{n+1} \left|\tfrac{x-y_{n,d}}{x-y_{n,k}}\right|^{\mu} |x-y_{n,k+m}|^p\, \omega\big(f^{(p)}; \tfrac{|x-y_{n,k+m}|}{m-p}\big).$$

For S_3 we remark that

$$\begin{aligned}|x-y_{n,k+m}|^p &\le (|x-y_{n,k}|+|y_{n,k+m}-y_{n,k}|)^p\\ &=\sum_{r=0}^{p}\tbinom{p}{r}\,|x-y_{n,k}|^{p-r}\,|y_{n,k+m}-y_{n,k}|^r\,.\end{aligned}$$

Now, we have

$$\begin{aligned}S_3 &\le |x-y_{n,d}|^\mu \sum_{k=d+1}^{n+1}\sum_{r=0}^{p}\tbinom{p}{r}\frac{\left|y_{n,k}-y_{n,k+m}\right|^r}{\left|x-y_{n,k}\right|^{\mu-p+r}}\ \omega\big(f^{(p)};\tfrac{|x-y_{n,k+m}|}{m-p}\big)\\ &\le \pi^p\big(\tfrac{\sqrt{1-x^2}}{n}\big)^p\left(\sum_{r=0}^{p}\tbinom{p}{r}m^r\right)\sum_{k=d+1}^{n+1}\frac{1}{|k-d|^{\mu-p}}\ \omega\big(f^{(p)};\tfrac{|x-y_{n,k+m}|}{m-p}\big)\\ &=\big(1+\tfrac{1}{m-p}\big)\pi^p(m+1)^p\big(\tfrac{\sqrt{1-x^2}}{n}\big)^p\sum_{k=d+1}^{n+1}\frac{1}{|k-d|^{\mu-p}}\ \omega\big(f^{(p)};|x-y_{n,k+m}|\,\big).\end{aligned}$$

We remark that for $\mu-p>1$, $\sum_{k=d+1}^{n+1}|k-d|^{p-\mu}$ is bounded by M.

From this observation and (10), (7) we obtain

$$\begin{aligned}&\omega\big(f^{(p)};|x-y_{n,k+m}|\,\big)\le\\ &\le M(m\pi+1)\omega\big(f^{(p)};\tfrac{\sqrt{1-x^2}}{n}\big)+2(\pi+1)\omega\big(f^{(p)};\tfrac{|k-d|}{n}\sqrt{1-x^2}\big).\end{aligned}$$

In conclusion, we obtain for S_3 the following inequality

$$S_3\le\big(\tfrac{\sqrt{1-x^2}}{n}\big)^p\left[c_3\omega\big(f^{(p)};\tfrac{\sqrt{1-x^2}}{n}\big)+c_4\sum_{k=d+1}^{n+1}\frac{\omega\big(f^{(p)};\frac{|k-d|}{n}\sqrt{1-x^2}\big)}{|k-d|^{\mu-p}}\right]$$

with the constants

$$\begin{aligned}c_3&=\pi^p(m+1)^pM(m\pi+1)\big(1+\tfrac{1}{m-p}\big),\\ c_4&=2\pi^p(m+1)^p(\pi+1)\big(1+\tfrac{1}{m-p}\big).\end{aligned}$$

Since

$$\omega\big(f^{(p)};\tfrac{\sqrt{1-x^2}}{n}\big)\le\tfrac{1}{n^{\mu-p-1}}\int_{n^{-1}}^{1}\frac{\omega\big(f^{(p)};t\sqrt{1-x^2}\big)}{t^{\mu-p}}\mathrm{d}t$$

and

$$\begin{aligned}&S_1+S_2+S_3\le\\ &\le\big(\tfrac{\sqrt{1-x^2}}{n}\big)^p\Big[C(c_2,c_3)\cdot\omega\big(f^{(p)};\tfrac{\sqrt{1-x^2}}{n}\big)+C(c_1,c_4)\cdot\omega\big(f^{(p)};\tfrac{|k-d|}{n}\sqrt{1-x^2}\big)\Big]\end{aligned}$$

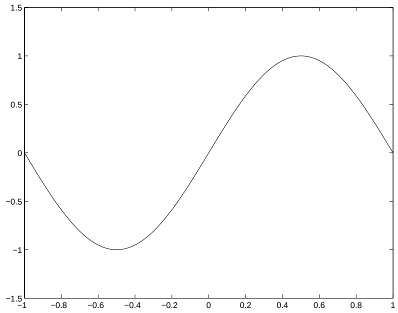
(3) follows. □

3. EXAMPLES AND GRAPHS

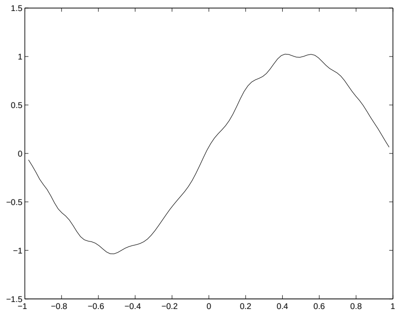
Let us consider the function $f : [-1, 1] \to \mathbb{R}$,

$$f(x) = \sin \pi x.$$

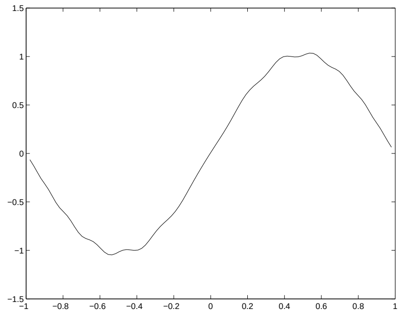
Its graph appears in figure 1, together with the Shepard-Lagrange approximation functions for $\mu \in \{2, 4\}$, $n = 16$ and $m \in \{1, 2\}$.



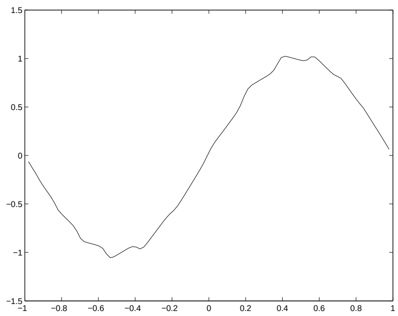
(a) Graph of f.



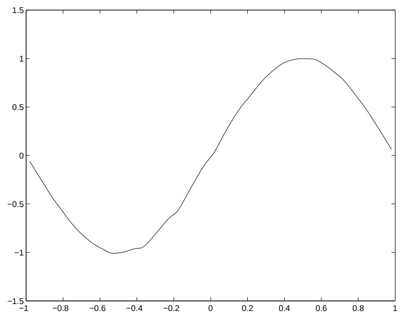
(b) Shepard-Lagrange, $n = 16$, $\mu = 2$, $m = 1$.



(c) Shepard-Lagrange, $n = 16$, $\mu = 2$, $m = 2$.



(d) Shepard-Lagrange, $n = 16$, $\mu = 4$, $m = 1$.



(e) Shepard-Lagrange, $n = 16$, $\mu = 4$, $m = 2$.

Fig. 1. Graph of f and various Shepard-Lagrange interpolants.

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