SELF-SIMILAR SETS IN CONVEX METRIC SPACES

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. The purpose of this paper is to present some existence and uniqueness results for self-similar sets in convex complete metric spaces.


Keywords. Self-similar set, generalized contraction, convex metric space.

1. INTRODUCTION

Let $(X,d)$ be a complete metric space and $f_i$, $i \in \{1, \ldots, m\}$ be single-valued mappings of $X$ into itself. Let $P_{cp}(X)$ be the space of all nonempty compact subsets of $X$ and denote by $H$ the Hausdorff-Pompeiu metric on $P_{cp}(X)$. If we define the operator $T : (P_{cp}(X), H) \to (P_{cp}(X), H)$ by the formula $T(Y) = \bigcup_{i=1}^{m} f_i(Y)$, then $T$ is called the fractal operator generated by the so-called iterated functions system $f = \{f_1, f_2, \ldots, f_m\}$. Any fixed point of $T$ is, by definition, a self-similar set for the iterated functions system $f = \{f_1, f_2, \ldots, f_m\}$. It is known that, in many cases, the Hausdorff dimension of a self-similar is not an integer. For this reason, in all these cases self-similar sets are fractals and $P_{cp}(X)$ is the space of fractals. Moreover, self-similar sets among the fractals form an important class, since many of them have computable Hausdorff dimensions. Regarding the existence of self-similar sets, if $f_i$ are $\alpha_i$-contractions for $i \in \{1, \ldots, m\}$ then the operator $T$ is a $\max(\alpha_i \mid i \in \{1, \ldots, m\})$-contraction having a unique fixed point. Hence the iterated functions system $f = \{f_1, f_2, \ldots, f_m\}$ has a unique self-similar set (this is a result of Hutchinson and Barnsley, see for example Petrușel [9], Yamaguti, Hata, Kigami [10]). Same conclusion holds, one hand when $f_i$, $i \in \{1, \ldots, m\}$ are $\varphi$-contractions (see I.A.Rus [11]) and on the other hand when $f_i$, $i \in \{1, \ldots, m\}$ are Meir-Keeler type operators (see Petrușel [7], [10]).

The purpose of this paper is to study the existence and uniqueness of self-similar sets for iterated functions systems on convex complete metric spaces. The multi-valued case is also considered.

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2. MAIN RESULTS

Let \((X,d)\) be a metric space and \(f : X \to X\) be a single-valued operator. We will consider first some contraction-type conditions for the operator \(f\).

**Definition 1.** The operator \(f : X \to X\) is said to be:

i) \(\alpha\)-contraction if \(\alpha \in [0,1]\) and \(d(f(x), f(y)) \leq \alpha d(x,y)\), for each \(x, y \in X\).

ii) strict contraction if \(d(f(x), f(y)) < d(x,y)\), for each \(x, y \in X\), with \(x \neq y\).

iii) Meir-Keeler type operator if for each \(\eta > 0\) there exists \(\delta > 0\) such that for each \(x, y \in X\) with \(\eta \leq d(x,y) < \eta + \delta\) it follows \(d(f(x), f(y)) < \eta\).

(iv) Matkowski-Wegrzyk type operator if for each \(\eta > 0\) there is \(\delta > 0\) such that for each \(x, y \in X\) with \(\eta < d(x,y) < \eta + \delta\) it follows \(d(f(x), f(y)) \leq \eta\).

(v) Boyd-Wong type operator if for each \(x, y \in X\) we have \(d(f(x), f(y)) \leq k(d(x,y))\), where \(k : [0, \infty) \to [0, \infty)\) is a function satisfying the property \(k(t) < t\), for each \(t > 0\).

(vi) Rakotch type operator if \(d(f(x), f(y)) \leq k(d(x,y))d(x,y)\), for each \(x, y \in X\), where \(k : [0, \infty) \to [0, 1)\) is a nonincreasing function with the property \(k(t) < t\), for each \(t > 0\).

Let us observe that, the condition (i) implies (ii), (i) implies (iii), (iii) implies each of the conditions (ii) (iv), (v) and (vi). Jachymski (see [2]) proved that the reverse implications, i.e. (iv) implies (iii) and (iv) implies (ii), are, in general, not true.

**Definition 2.** A metric space is said to be metrically convex if for every distinct points \(x, y \in X\) there exists \(z \in X\) such that \(d(x,y) = d(x,z) + d(z,y)\) and \(x \neq z \neq y\).

For example, every normalized space and any of its convex subsets is metrically convex. An important property of such spaces is:

**Theorem 3.** If \((X,d)\) is a complete and metrically convex metric space, then for every distinct points \(x, y \in X\) and for each \(\lambda \in [0,1]\) there exists \(z \in X\) such that \(d(z,x) = \lambda d(x,y)\) and \(d(z,y) = (1-\lambda)d(x,y)\).

In what follows we use the terminology “convex metric space” for a metrically convex metric space.

The following result is proved by Petruşel [7].

**Theorem 4.** Let \((X,d)\) be a complete metric space and \(f_i : X \to X, i \in \{1,2, \ldots, m\}\) be a finite family of Meir-Keeler type operators. Then:

a) the fractal operator \(T : (P_{cp}(X), H) \to (P_{cp}(X), H)\) generated by the iterated functions system \(f = \{f_1, f_2, \ldots, f_m\}\) is a Meir-Keller type operator.
b) the iterated function system $f = \{f_1, f_2, ..., f_m\}$ has a unique self-similar set $A^*$, having the property that for each compact subset $A_0$ of $X$ the sequence of successive approximations $(T^n(A_0))_{n \in \mathbb{N}}$ converges to $A^*$.

The following theorem gives us the equivalence of some generalized contractive conditions (see Matkowski and Wegrzyk [4]):

**Theorem 5.** Let $(X, d)$ be a convex complete metric space and $(Y, \rho)$ be a metric space. Let $f : X \to Y$ be an arbitrary function. Then the following assertions are equivalent:

(i) $f$ is a Meir-Keeler type operator;

(ii) $f$ is a Rakotch type operator;

(iii) $f$ is a Boyd-Wong type operator;

(iv) $f$ is a Matkowski-Wegrzyk type operator.

Moreover, if $f$ fulfills one of the above conditions, then the function $k$ in (iii) is strictly increasing, concave and continuously differentiable in $[0, \infty]$ and the function $k$ in (ii) is continuous.

Using the above result, we obtain the first main result of this paper:

**Theorem 6.** Let $(X, d)$ be a convex complete metric space and $f_i : X \to X$, $i \in \{1, 2, ..., m\}$ be a finite family of Matkowski-Wegrzyk type operators. Then the iterated functions system $f = \{f_1, f_2, ..., f_m\}$ has a unique self-similar set $A^*$. Moreover, for each compact subset $A_0$ of $X$ the sequence of successive approximations $(T^n(A_0))_{n \in \mathbb{N}}$ converges to $A^*$.

**Proof.** From Theorem 5 we have that $f = \{f_1, f_2, ..., f_m\}$ is an iterated function system having the property that each function $f_i$ satisfies to a contraction type condition. From the classical result of Hutchinson and Barnsley we obtain that the fractal operator $T$ is a contraction too from $P_{cp}(X)$ to itself. Hence, by Banach contraction principle we get the desired conclusion. The proof is complete.

Let us remark that the above theorem can be proved, via Theorem 5, using Theorem 4 instead of Banach contraction principle.

**Remark.** The similarity dimension $d$ of a self-similar set $A^*$ corresponding to an iterated functions system $f = \{f_1, f_2, ..., f_m\}$, where $f_i$ is an $\alpha_i$-contraction, for each $i \in \{1, 2, ..., m\}$, is defined as the unique positive root of the equation $\sum_{i=1}^{m} \alpha_i^d = 1$. It is easy to see now that the similarity dimension of a self-similar set generated by a finite family of Matkowski-Wegrzyk type operators on a convex complete metric space can be calculated in the same way.

Let us consider now the multi-valued case. Let $F_1, \ldots, F_m : X \to P_{cp}(X)$ be a finite family of upper semi-continuous (briefly u.s.c.) multi-valued operators. We define the multi-fractal operator $T_F$ generated by the following iterated...
multi-functions system  \( F = (F_1, F_2, \ldots, F_m) \), by the following relation:  \( T_F : P_{cp}(X) \to P_{cp}(X) \),  \( T_F(Y) = \bigcup_{i=1}^{m} F_i(Y) \). In this framework, a nonempty compact subset  \( A^* \) of  \( X \) is said to be a self-similar set for the iterated multi-functions system  \( F = (F_1, F_2, \ldots, F_m) \) if and only if it is a fixed point for the associated multi-fractal operator. The following notions are needed in the sequel.

**Definition 7.** The multi-valued operator  \( F : X \to P_{cp}(X) \) is said to be:

i) **multi-valued  \( \alpha \)-contraction** if  \( \alpha \in [0, 1[ \) and for each  \( x, y \in X \) we have  \( wH(F(x), F(y)) \leq \alpha d(x, y) \).

ii) **multi-valued strict contraction** if  \( H(F(x), F(y)) < d(x, y) \), for each  \( x, y \in X \), with  \( x \neq y \).

iii) **multi-valued Meir-Keeler type operator** if for each  \( \eta > 0 \) there exists  \( \delta > 0 \) such that for each  \( x, y \in X \) with  \( \eta \leq d(x, y) < \eta + \delta \) it follows  \( H(F(x), F(y)) < \eta \).

iv) **multi-valued Matkowski-Wegrzyk type operator** if for each  \( \eta > 0 \) there is  \( \delta > 0 \) such that for  \( x, y \in X \) with  \( \eta < d(x, y) < \eta + \delta \) it follows  \( H(F(x), F(y)) < \eta \).

v) **multi-valued Rakotch type operator** if for each  \( x, y \in X \) we have  \( H(F(x), F(y)) \leq k(d(x, y)) \), where  \( k : [0, \infty) \to [0, \infty) \) is a function satisfying the property  \( k(t) < t \), for each  \( t > 0 \).

vi) **multi-valued Boyd-Wong type operator** if for each  \( x, y \in X \) we have  \( H(F(x), F(y)) \leq k(d(x, y))d(x, y) \), where  \( k : [0, \infty) \to [0, 1) \) is a function with the property  \( k(t) < t \), for each  \( t > 0 \).

A similar discussion with the single-valued case can be done also for the multi-valued setting.

Regarding the existence and uniqueness of self-similar sets for iterated multi-functions systems, it is well-known that a finite family of multi-valued contractions has an unique self-similar set. Moreover, for the case of multi-valued Meir-Keeler operators the following result holds (see Petrușel [7]):

**Theorem 8.** Let  \( (X, d) \) be a complete metric space and  \( F_i : X \to P_{cp}(X) \),  \( i \in \{1, \ldots, m\} \) be a finite family of multi-valued Meir-Keeler type operators. Then:

a) the multi-fractal operator  \( T_F : P_{cp}(X) \to P_{cp}(X) \) is a Meir-Keeler type operator.

b) the iterated multi-functions system  \( F = (F_1, F_2, \ldots, F_m) \) has a unique self-similar set  \( A^* \). Moreover, for each compact subset  \( A_0 \) of  \( X \) the sequence of successive approximations  \( (T_F^n(A_0))_{n \in \mathbb{N}} \) converges to  \( A^* \).

The equivalence between the following generalized contractions conditions is proved in Moț [5]:

**Theorem 9.** Let  \( (X, d) \) be a convex complete metric space and  \( (Y, \rho) \) be a metric space. Let  \( F : X \to P_{cp}(Y) \) be multi-function. Then the following assertions are equivalent:
(i) $F$ is a multi-valued Meir-Keeler type operator;
(ii) $F$ is a multi-valued Rakotch type operator;
(iii) $F$ is a multi-valued Boyd-Wong type operator;
(iv) $F$ is a multi-valued Matkowski-Wegrzyk type operator.

From Theorem 8 and Theorem 9 we obtain:

**Theorem 10.** Let $(X,d)$ be a convex complete metric space and $F_i : X \rightarrow P_{cp}(X), i \in \{1,2,\ldots,m\}$ be a finite family of multi-valued Matkowski-Wegrzyk type operators. Then the iterated multi-functions system $F = \{F_1,F_2,\ldots,F_m\}$ has a unique self-similar set $A^*$. Moreover, for each compact subset $A_0$ of $X$ the sequence of successive approximations $(T^n(A_0))_{n \in \mathbb{N}}$ converges to $A^*$.

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Received by the editors: July 3, 2004.