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SELF-SIMILAR SETS IN CONVEX METRIC SPACES

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. The purpose of this paper is to present some existence and uniqueness results for self-similar sets in convex complete metric spaces.

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1. INTRODUCTION

Let (X,d) be a complete metric space and f_i, $i \in \{1,\ldots,m\}$ be single-valued mappings of X into itself. Let $P_{cp}(X)$ be the space of all nonempty compact subsets of X and denote by H the Hausdorff-Pompeiu metric on $P_{cp}(X)$. If we define the operator $T : (P_{cp}(X),H) \to (P_{cp}(X),H)$ by the formula $T(Y) = \bigcup_{i=1}^{m} f_i(Y)$, then T is called the fractal operator generated by the so-called iterated functions system $f = \{f_1, f_2, ..., f_m\}$. Any fixed point of T is, by definition, a self-similar set for the iterated functions system $f = \{f_1, f_2, ..., f_m\}$. It is known that, in many cases, the Hausdorff dimension of a self-similar is not an integer. For this reason, in all these cases self-similar sets are fractals and $P_{cp}(X)$ is the space of fractals. Moreover, self-similar sets among the fractals form an important class, since many of them have computable Hausdorff dimensions. Regarding the existence of self-similar sets, if f_i are α_i-contractions for $i \in \{1,\ldots,m\}$ then the operator T is a $\max(\alpha_i |\ i \in \{1,\ldots,m\})$-contraction having a unique fixed point. Hence the iterated functions system $f = \{f_1, f_2, ..., f_m\}$ has a unique self-similar set (this is a result of Hutchinson and Barnsley, see for example Petruşel [9], Yamaguti, Hata, Kigami [13]). Same conclusion holds, one hand when f_i, $i \in \{1,\ldots,m\}$ are φ-contractions (see I.A.Rus [11]) and on the other hand when f_i, $i \in \{1,\ldots,m\}$ are Meir-Keeler type operators (see Petruşel [7], [10]).

The purpose of this paper is to study the existence and uniqueness of self-similar sets for iterated functions systems on convex complete metric spaces. The multi-valued case is also considered.

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2. MAIN RESULTS

Let (X, d) be a metric space and $f : X \to X$ be a single-valued operator. We will consider first some contraction-type conditions for the operator f.

Definition 1. *The operator $f : X \to X$ is said to be:*

i) α**-contraction** *if $\alpha \in [0, 1[$ and $d(f(x), f(y)) \leq \alpha d(x, y)$, for each $x, y \in X$.*

ii) **strict contraction** *if $d(f(x), f(y)) < d(x, y)$, for each $x, y \in X$, with $x \neq y$.*

iii) **Meir-Keeler type operator** *if for each $\eta > 0$ there exists $\delta > 0$ such that for each $x, y \in X$ with $\eta \leq d(x, y) < \eta + \delta$ it follows $d(f(x), f(y)) < \eta$.*

(iv) **Matkowski-Wegrzyk type operator** *if for each for $\eta > 0$ there is $\delta > 0$ such that for $x, y \in X$, with $\eta < d(x, y) < \eta + \delta$ it follows $d(f(x), f(y)) \leq \eta$.*

(v) **Boyd-Wong type operator** *if for each $x, y \in X$ we have $d(f(x), f(y)) \leq k(d(x, y))$, where $k : [0, \infty) \to [0, \infty)$ is a function satisfying the property $k(t) < t$, for each $t > 0$.*

(vi) **Rakotch type operator** *if $d(f(x), f(y)) \leq k(d(x, y))d(x, y)$, for each $x, y \in X$, where $k : [0, \infty) \to [0, 1)$ is a nonincreasing function with the property $k(t) < t$, for each $t > 0$.*

Let us observe that, the condition (i) implies (ii), (i) implies (iii), (iii) implies each of the conditions (ii) (iv), (v) and (vi). Jachymski (see [2]) proved that the reverse implications, i.e. (iv) implies (iii) and (iv) implies (ii), are, in general, not true.

Definition 2. *A metric space is said to be metrically convex if for every distinct points $x, y \in X$ there exists $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$ and $x \neq z \neq y$.*

For example, every normalized space and any of its convex subsets is metrically convex. An important property of such spaces is:

Theorem 3. *If (X, d) is a complete and metrically convex metric space, then for every distinct points $x, y \in X$ and for each $\lambda \in]0, 1[$ there exists $z \in X$ such that $d(z, x) = \lambda d(x, y)$ and $d(z, y) = (1 - \lambda)d(x, y)$.*

In what follows we use the terminology "convex metric space" for a metrically convex metric space.

The following result is proved by Petruşel [7].

Theorem 4. *Let (X, d) be a complete metric space and $f_i : X \to X$, $i \in \{1, 2, \ldots, m\}$ be a finite family of Meir-Keeler type operators. Then:*

a) *the fractal operator $T : (P_{cp}(X), H) \to (P_{cp}(X), H)$ generated by the iterated functions system $f = \{f_1, f_2, ..., f_m\}$ is a Meir-Keller type operator.*

b) *the iterated function system* $f = \{f_1, f_2, ..., f_m\}$ *has a unique self-similar set* A^*, *having the property that for each compact subset* A_0 *of* X *the sequence of successive approximations* $(T^n(A_0))_{n\in N}$ *converges to* A^*.

The following theorem gives us the equivalence of some generalized contractions conditions (see Matkowski and Wegrzyk [4]):

THEOREM 5. *Let* (X, d) *be a convex complete metric space and* (Y, ρ) *be a metric space. Let* $f : X \to Y$ *be an arbitrary function. Then the following assertions are equivalent:*

(i) f *is a Meir-Keeler type operator;*
(ii) f *is a Rakotch type operator;*
(iii) f *is a Boyd-Wong type operator;*
(iv) f *is a Matkowski-Wegrzyk type operator.*

Moreover, if f *fulfils one of the above conditions, then the function* k *in (iii) is strictly increasing, concave and continuously differentiable in* $[0, \infty[$ *and the function* k *in (ii) is continuous.*

Using the above result, we obtain the first main result of this paper:

THEOREM 6. *Let* (X, d) *be a convex complete metric space and* $f_i : X \to X$, $i \in \{1, 2, \dots, m\}$ *be a finite family of Matkowski-Wegrzyk type operators. Then the iterated functions system* $f = \{f_1, f_2, ..., f_m\}$ *has a unique self-similar set* A^*. *Moreover, for each compact subset* A_0 *of* X *the sequence of successive approximations* $(T^n(A_0))_{n\in N}$ *converges to* A^*.

Proof. From Theorem 5 we have that $f = \{f_1, f_2, ..., f_m\}$ is an iterated function system having the property that each function f_i satisfies to a contraction type condition. From the classical result of Hutchinson and Barnsley we obtain that the fractal operator T is a contraction too from $P_{cp}(X)$ to itself. Hence, by Banach contraction principle we get the desired conclusion. The proof is complete. □

Let us remark that the above theorem can be proved, via Theorem 5, using Theorem 4 instead of Banach contraction principle.

REMARK. The similarity dimension d of a self-similar set A^* corresponding to an iterated functions system $f = \{f_1, f_2, ..., f_m\}$, where f_i is an α_i-contraction, for each $i \in \{1, 2, ..., m\}$, is defined as the unique positive root of the equation $\sum_{i=1}^{m} \alpha_i^d = 1$. It is easy to see now that the similarity dimension of a self-similar set generated by a finite family of Matkowski-Wegrzyk type operators on a convex complete metric space can be calculated in the same way. □

Let us consider now the multi-valued case. Let $F_1, \dots, F_m : X \to P_{cp}(X)$ be a finite family of upper semi-continuous (briefly u.s.c.) multi-valued operators. We define the multi-fractal operator T_F generated by the following iterated multi-functions system $F = (F_1, F_2, \ldots, F_m)$, by the following relation: $T_F : P_{cp}(X) \to P_{cp}(X)$, $\quad T_F(Y) = \bigcup_{i=1}^{m} F_i(Y)$. In this framework, a nonempty compact subset A^* of X is said to be a self-similar set for the iterated multi-functions system $F = (F_1, F_2, \ldots, F_m)$ if and only if it is a fixed point for the associated multi-fractal operator. The following notions are needed in the sequel.

DEFINITION 7. *The multi-valued operator $F : X \to P_{cp}(X)$ is said to be:*

i) **multi-valued α-contraction** *if $\alpha \in [0, 1[$ and for each $x, y \in X$ we have $wH(F(x), F(y)) \leq \alpha d(x, y)$.*
ii) **multi-valued strict contraction** *if $H(F(x), F(y)) < d(x, y)$, for each $x, y \in X$, with $x \neq y$.*
iii) **multi-valued Meir-Keeler type operator** *if for each $\eta > 0$ there exists $\delta > 0$ such that for each $x, y \in X$ with $\eta \leq d(x, y) < \eta + \delta$ it follows $H(F(x), F(y)) < \eta$.*
(iv) **multi-valued Matkowski-Wegrzyk type operator** *if for each for $\eta > 0$ there is $\delta > 0$ such that for $x, y \in X$, with $\eta < d(x, y) < \eta + \delta$ it follows $H(F(x), F(y)) \leq \eta$.*
(v) **multi-valued Rakotch type operator** *if for each $x, y \in X$ we have $H(F(x), F(y)) \leq k(d(x, y))$, where $k : [0, \infty) \to [0, \infty)$ is a function satisfying the property $k(t) < t$, for each $t > 0$.*
(vi) **multi-valued Boyd-Wong type operator** *if for each $x, y \in X$ we have $H(F(x), F(y)) \leq k(d(x, y))d(x, y)$, where $k : [0, \infty) \to [0, 1)$ is a function with the property $k(t) < t$, for each $t > 0$.*

A similar discussion with the single-valued case can be done also for the multi-valued setting.

Regarding the existence and uniqueness of self-similar sets for iterated multi-functions systems, it is well-known that a finite family of multi-valued contractions has an unique self-similar set. Moreover, for the case of multi-valued Meir-Keeler operators the following result holds (see Petruşel [7]):

THEOREM 8. *Let (X, d) be a complete metric space and $F_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be a finite family of multi-valued Meir-Keeler type operators. Then:*

a) *the multi-fractal operator $T_F : P_{cp}(X) \to P_{cp}(X)$ is a Meir-Keeler type operator.*
b) *the iterated multi-functions system $F = (F_1, F_2, \ldots, F_m)$ has a unique self-similar set A^*. Moreover, for each compact subset A_0 of X the sequence of successive approximations $(T_F^n(A_0))_{n \in N}$ converges to A^*.*

The equivalence between the following generalized contractions conditions is proved in Moţ [5]:

THEOREM 9. *Let (X, d) be a convex complete metric space and (Y, ρ) be a metric space. Let $F : X \to P_{cp}(Y)$ be multi-function. Then the following assertions are equivalent:*

(i) *F is a multi-valued Meir-Keeler type operator;*
(ii) *F is a multi-valued Rakotch type operator;*
(iii) *F is a multi-valued Boyd-Wong type operator;*
(iv) *F is a multi-valued Matkowski-Wegrzyk type operator.*

From Theorem 8 and Theorem 9 we obtain:

Theorem 10. *Let (X, d) be a convex complete metric space and $F_i : X \to P_{cp}(X)$, $i \in \{1, 2, \ldots, m\}$ be a finite family of multi-valued Matkowski-Wegrzyk type operators. Then the iterated multi- functions system $F = \{F_1, F_2, \ldots, F_m\}$ has a unique self-similar set A^*. Moreover, for each compact subset A_0 of X the sequence of successive approximations $(T^n(A_0))_{n\in N}$ converges to A^*.*

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